Astronomical and Astrophysical Transactions (*AApTr*), 2025, Vol. 35, Issue 1, pp. 199–208,
ISSN 1055-6796, Photocopying permitted by license only, ©Cambridge Scientific Publishers

Three-dimensional potentials with local integrals of motion: modeling stellar systems with dark matter

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Received 27 April, 2025; accepted 31 May, 2025

A generalized expression for potential energy in three-dimensional space is investigated, which allows the existence of a local integral of motion in rotating systems. A form of potential energy is introduced that includes contributions from the gradient of a scalar field, an external potential, and rotational effects. This approach enables the description of effective potentials in galactic dynamics, including contributions from both ordinary and dark matter, as well as rotational effects.

Examples of applications are discussed, ranging from analytical models of dark matter halos to hypothetical objects such as self-gravitating Bose–Einstein condensates of dark matter. It is shown that the proposed form of the potential is analogous to expressions found in the Lagrangians and energy functionals of Fuzzy Dark Matter (FDM) and Scalar Field Dark Matter (SFDM) models. This allows for the interpretation of observed galaxy properties, such as flat rotation curves, through the contributions of dark matter and kinetic-gravitational interactions in scalar fields.

Keywords: Stellar systems, generalized three-dimensional potential, local integral of motion, dark matter

1 Introduction

In stellar astronomy and astrophysics, a key element of studying the dynamics of various cosmic objects is the precise description of the potential fields that influence their motion. Of particular importance is the understanding of potential energy in systems where not only gravitational interactions are present but also effects related to rotation or the presence of dark matter. Models describing such systems must account for both gravitational forces from visible matter and contributions from invisible matter, including hypothetical forms of dark matter.

Traditionally, potential functions are used to describe the dynamics of such systems, and they may include various components such as kinetic energy, gravitational potential, and rotational effects. However, in certain cases, such as in galaxy models

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or gravitating scalar fields, it is important to consider local integrals of motion, which can significantly simplify the mathematical description and physical interpretation of the system.

This work examines a generalized expression for potential derived in the studies of (Antonov & Shamshiev, 1992; Shamshiev, 2022), which enables the existence of a local integral of motion in three-dimensional space. The expression introduced encompasses several key components: the gradients of a scalar field, contributions from an external potential, and rotational effects related to the angular velocity of the system. This approach allows for an effective description of potential fields arising in models of galaxies, dark-matter halos, and self-gravitating objects such as Bose–Einstein condensates (BEC), which can be used to interpret observed astronomical phenomena, such as galaxy rotation curves (Chavanis, 2011; Ureña-López et al., 2002).

The work focuses on an analytical approach to the calculation of effective potentials, where each component of the potential (kinetic, rotational, and gravitational) can be clearly interpreted in the context of physical processes occurring in the system. In particular, models are considered in which potential energy is a function of distributed mass, including contributions from dark matter, which is particularly relevant for explaining observed anomalies in galaxy dynamics.

The proposed form of the potential in this work can serve as a foundation for further research in theoretical astrophysics and cosmology, especially in the context of studying the complex interrelations between visible and invisible matter in gravitationally bound systems.

2 General formulas

In Shamshiev (2022), a steady-state stellar system rotating with a constant angular velocity Ω was considered. In this three-dimensional case, only the well-known Jacobi integral exists, which, in Cartesian coordinates, takes the form

$$\frac{u^2+v^2+w^2}{2}-\frac{\Omega^2(x^2+y^2)}{2}-U(x,y,z)=h, \tag{1}$$

where h is an arbitrary energy constant.

We now examine the expression for the potential that admits a local integral of motion in three-dimensional space, as introduced in the work Shamshiev (2022). This expression incorporates gradient terms, specific functions, and rotational effects

$$U(x,y,z)=\frac{1}{2}\left[(\nabla\Phi)^2+U_0(L)+\Omega\left(y\frac{\partial\Phi}{\partial x}-x\frac{\partial\Phi}{\partial y}\right)\right], \tag{2}$$

where $\Phi=\Phi(x,y,z)$ is an arbitrary scalar function describing the potential field, Ω is the angular velocity responsible for the contribution of Coriolis or centrifugal forces, and $U_0(L)$ is some function of L, which could represent an external potential or the contribution from another interaction. In principle, $U_0(L)$ may describe the contribution of dark matter, which influences the motion of matter or the structure of the potential. It should be noted that the function L must satisfy the condition

$$\left(\frac{\partial L}{\partial x}\right)^2+\left(\frac{\partial L}{\partial y}\right)^2+\left(\frac{\partial L}{\partial z}\right)^2=1, \tag{3}$$

which formally coincides with the eikonal equation, which is well known in geometric optics (Landau & Lifshits, 2018). The square of the gradient of the potential, commonly encountered in energy expressions, corresponds to the kinetic energy:

$$(\nabla\Phi)^2 = |\nabla\Phi|^2 = \left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2. \tag{4}$$

The third term $\left(y\frac{\partial\Phi}{\partial x} - x\frac{\partial\Phi}{\partial y}\right)$ is the z-component of the rotational moment or the orbital angular momentum.

In models of galaxies, clusters, or rotating systems, an effective potential is often introduced, which includes the following:

- Kinetic energy or contribution from the field (e.g., $(\nabla\Phi)^2$),
- Contribution from the rotation of the system (through angular momentum or Coriolis terms),
- Gravitational potential from visible and dark matter.

In such cases, $U_0(L)$ may be a function of the distance $L = \sqrt{x^2 + y^2 + z^2}$, and it would describe the gravitational potential of the distributed mass, including invisible matter, i.e. dark matter.

3 Examples of the application of the potential in gravitationally bound system

Example 1: Effective potential of dark matter in galactic dynamics with rotation

Let us consider a model of a spherical dark matter halo described by the Plummer (1911) potential (Binney & Tremaine (2008)),

$$U_0(r) = -\frac{GM}{\sqrt{r^2 + r_c^2}}, \tag{5}$$

where G is the gravitational constant, M is the mass of the halo, and r_c is the core (characteristic) radius. This potential corresponds to the following density distribution:

$$\rho(r) = \frac{3M}{4\pi r_c^3}\left(1 + \frac{r^2}{r_c^2}\right)^{-5/2}. \tag{6}$$

To take the galaxy's rotation into account, we include the rotational contribution to the effective potential:

$$U_{\rm rot}(x, y) = \Omega\left(y\frac{\partial\Phi}{\partial x} - x\frac{\partial\Phi}{\partial y}\right), \tag{7}$$

which is important when modeling disk-like structures.

The total effective potential of the system becomes:

$$U_{\rm eff}(x, y, z) = \frac{1}{2}|\nabla\Phi|^2 + U_0(r) + U_{\rm rot}(x, y). \tag{8}$$

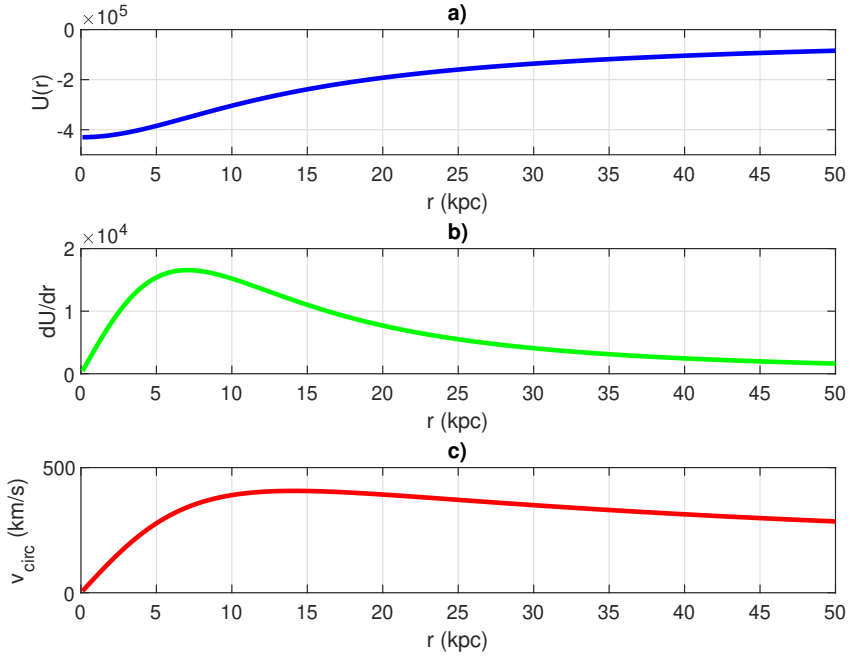


Figure 1 a) Potential $U(r)$, b) gravitational acceleration $\dfrac{dU}{dr}$, c) circular velocity $v_{\rm circ}(r)$.

For our models, we adopt the following parameters: a halo mass of $M = 10^{12} M_{\odot}$, a core radius of $r_c = 10$ kpc, a radial range of $r \in [0.1, 50]$ kpc, and a galactic angular velocity of $\Omega = 25$ km/s/kpc.

Figure 1 shows the plots of the potential $U(r)$, its derivative $dU(r)/dr$, and the circular velocity $v_{\rm circ}(r) = \sqrt{r\, dU(r)/dr}$, which can be compared to observed rotation curves of galaxies (e.g., NGC 3198, Begeman (1989); de Blok et al. (2001)).

The first plot in Figure 1(a) shows how the gravitational potential changes with distance from the center. It is always negative (indicating attraction), reaches its minimum at the center, and gradually increases (i.e., becomes less negative) as r increases. The second plot in Figure 1(b) illustrates how the gravitational force varies with radial distance. The force increases with r near the center, reaches a maximum around $r \sim r_c$, and then decreases as the distance grows. The third plot in Figure 1(c) corresponds to the galaxy rotation curve in the halo model. It shows how the rotational velocity of stars and gas changes with distance from the center. Similar curves are observed in real galaxies, although they often exhibit more complex features, such as nearly flat rotation profiles at large radii.

This approach extends classical models by incorporating a physically motivated density distribution and rotational effects, which are particularly important in the context of relativistic halos, rotating Bose condensates, or scalar fields (e.g., fuzzy dark matter). The inclusion of $U_{\rm rot}$ makes the model relevant for unstable and axisymmetric disks. The model can be used as a test function for studying orbital stability, accretion structures, and the large-scale structure of the Universe.

Example 2. Modeling of gravitationally bound objects with dark matter and Bose–Einstein condensate

Consider an example where expression (2) is applied to model gravitationally bound objects, such as galaxies with dark matter or a self-gravitating Bose–Einstein Condensate (BEC) – a hypothetical model of dark matter. In some dark matter theories, it is assumed that dark matter consists of ultralight bosons (with masses of the order of 10^{-22} eV), which at low temperatures can form a gravitational Bose–Einstein condensate (Chavanis (2011); Ureña-López et al. (2002)). Such models are known as

- Fuzzy Dark Matter (FDM);
- Scalar Field Dark Matter (SFDM);
- BECDM (Bose–Einstein Condensate Dark Matter).

Then, how does the potential look? In such models, the equation for a stationary state is often derived from the energy functional:

$$E[\Phi] = \int \left[\frac{\hbar^2}{2m^2} |\nabla \Phi|^2 + U_0(|r|)|\Phi|^2 + \text{interaction/self-gravity} \right] d^3x. \tag{9}$$

However, if we consider a rotating system, a term of the form

$$U_{\text{rot}} = \Omega \left(y \frac{\partial \Phi}{\partial x} - x \frac{\partial \Phi}{\partial y} \right) \tag{10}$$

is added.

In this case, the potential energy density (or Lagrangian) will closely resemble our expression (2). What is $U_0(L)$ here? Usually, $U_0(L)$ is the gravitational potential of dark matter distributed in the form of a spherical halo, and it depends on the radius $L = |r|$.

In the BEC-halo example, its form can be derived from the Poisson equation:

$$\nabla^2 U_0(r) = 4\pi G m |\Phi(r)|^2, \tag{11}$$

that is, U_0 is self-consistently formed in response to the distribution of dark matter.

In this case, a typical form of the potential from a dark matter halo is:

$$U_0(r) = -\frac{GM(r)}{r}, \quad \text{where} \quad M(r) = -4\pi \int_0^r \rho_{\text{dm}}(r') r'^2 \, dr', \tag{12}$$

or in an approximate form, as we have already mentioned the potential of dark matter

$$U_0(r) = -v_0^2 \ln \left(1 + \frac{r^2}{r_c^2} \right). \tag{13}$$

A logarithmic dependence implies that at large distances, the growth of the potential slows down, yet it remains negative and tends toward zero, since the logarithmic function increases slowly. This is a typical case for dark matter in models where the dark matter potential decreases with increasing radius (Fig. 2).

In our numerical example, the potential gradient U_{grad} is defined as:

$$U_{\text{grad}} = \frac{1}{2} \left(\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right). \tag{14}$$

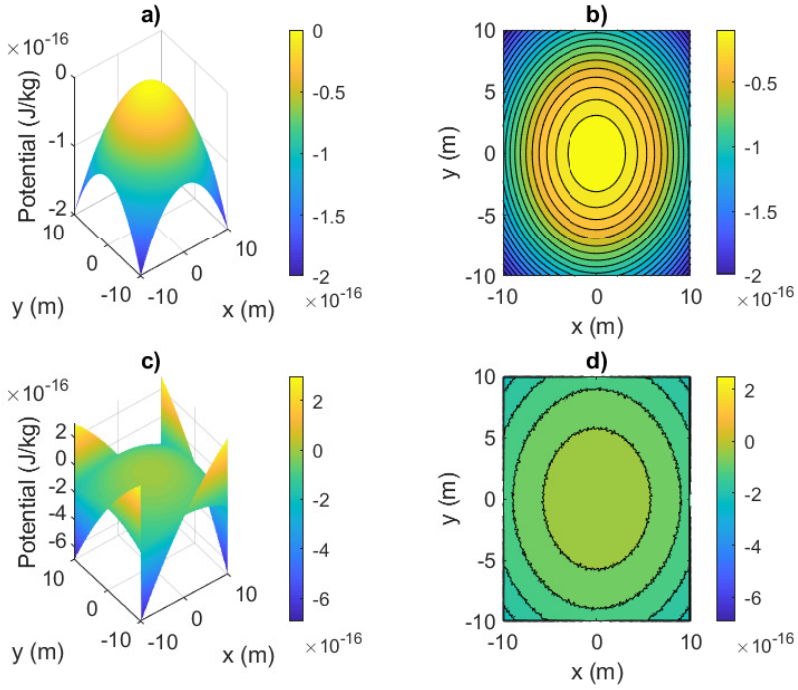


Figure 2 Visualization of gravitational and total potential components: (a) 3D surface of the gravitational potential U in the XY-plane; (b) contour plot of the gravitational potential U in the XY-plane; (c) 3D surface of the total potential $U_{\text{total}} = U_0 + U_{\text{grad}} + U_{\text{rot}}$; (d) contour plot of the total potential U_{total}.

This expression describes the contribution to the potential from the gradient of the scalar field, which is a Gaussian distribution. As r increases, the function Φ rapidly decreases (since it has the form of a Gaussian distribution), and its gradients also decrease. This means that the potential of the gradient also decreases with increasing distance from the center. For example, for the numerical calculation, we took

$$\Phi = \exp\left(-\frac{r^2}{2r_c^2}\right). \tag{15}$$

The plot in Figure 2(a) represents the shape of a modified gravitational potential resembling a logarithmic potential, which is commonly used in core-like dark matter halo models. The potential is most negative at the center ($R = 0$), where the mass is concentrated. As the radial distance increases, the potential becomes less negative, decreasing slowly in a logarithmic fashion. Such a potential profile is consistent with mass distributions found in ultralight dark matter halo models. Figure 2(b) shows the same potential as in panel (a), but in contour form. These contours represent equipotential lines. The nearly circular contours around the center indicate spherical symmetry in the mass distribution. The denser spacing of contours near the center implies a steeper gradient, corresponding to a stronger gravitational field in that region. Figure 2(c) illustrates the total gravitational potential, which becomes

asymmetric due to rotation. The contribution from U_{grad} acts as an additional pressure term that pushes mass outward. The U_{rot} term introduces anisotropy associated with the axis of rotation, leading to deformation in the overall potential structure. In Figure 2(d), the contours of the total potential U_{total} are no longer perfectly circular, in contrast to those in Figure 2(b). This deformation is analogous to the distortion of a gravitational well in a rotating system. As a result, particles are expected to follow non-circular orbits, particularly near the center. This behavior highlights the influence of both rotational and scalar field gradient contributions, causing the system to lose complete symmetry.

Thus, as the distance from the center of the system increases, the dominant contributions to the potential arise from the dark matter component and the gradient of the scalar field, both of which decrease with increasing r (Navarro et al. (1996)). Consequently, the total potential also decreases with distance from the center.

Example 3: Rotating galaxies with dark matter

Rotating spiral galaxies have nearly flat rotation curves, which requires the presence of dark matter. In models, the effective potential is used, where: gravitational contribution from the disk/stars: $U_{\text{vis}}(r)$, contribution from the dark halo: $U_0(r)$, centrifugal and Coriolis contributions: ΩL_z.

The total potential is written as:

$$U_{\text{eff}}(r) = U_{\text{vis}}(r) + U_0(r) - \frac{1}{2}\Omega^2 r^2, \tag{16}$$

or, more physically correctly, through the Lagrangian or energy functional. This corresponds to the balance between gravity and rotation, including the involvement of dark matter.

Figure 3 provides a means to understand the contribution of each galactic component to sustaining orbital motion, to compare the theoretical model with observed galactic rotation curves, and to confirm the necessity of dark matter at large radii, where the contribution of ordinary matter becomes negligible. By calculating the gravitational potentials and their corresponding derivatives for each component, the model quantitatively evaluates their influence on the rotational behavior of the galaxy across various radial distances.

The resulting rotation curve demonstrates the dominance of the central bulge at small radii, the stellar disk at intermediate radii, and dark matter at large distances. This approach offers valuable insight into the dynamical processes governing galactic rotation and emphasizes the distinct roles played by different galactic components. The visualization of the rotation curve serves as an effective tool for exploring the complex relationship between mass distribution and rotational dynamics in galaxies.

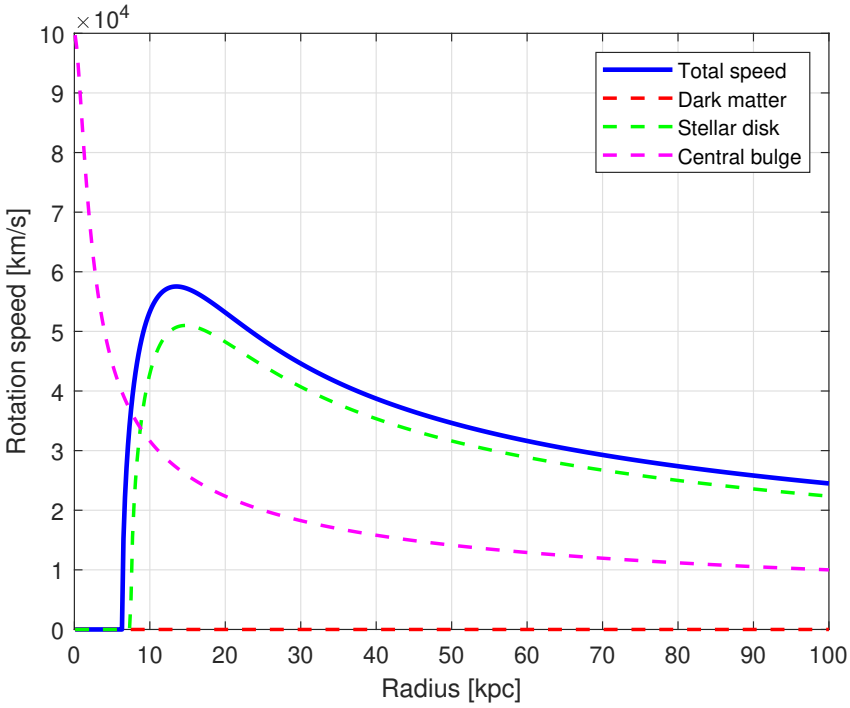


Figure 3 The effective potential $U_{dm}(r)$ as a function of the radius r.

4 Conclusion

Our three-dimensional potential (2), which admits a local integral of motion, represents a generalized form of the potential energy density, applicable to astrophysical systems involving dark matter. Within this model framework:

- $(\nabla\Phi)^2$ is interpreted as a kinetic contribution, incorporating – depending on the context – classical or quantum effects;
- $\Omega\left(y\dfrac{\partial\Phi}{\partial x} - x\dfrac{\partial\Phi}{\partial y}\right)$ describes the influence of rotation, including Coriolis and centrifugal effects;
- $U_0(L)$ models the gravitational contribution, including that from dark matter, within a spherically symmetric mass distribution.

The proposed potential form encompasses a wide range of applications, from analytic models of dark matter halos and stellar disks to modern approaches incorporating Bose–Einstein condensates and scalar fields. A key advantage is the existence of a local integral of motion, which simplifies the analysis of orbital dynamics and makes the model particularly useful for studying the stability of stellar trajectories and the structure of gravitationally bound systems.

The model can be adapted to construct realistic density profiles and rotation curves comparable with observations of spiral galaxies. This makes it a valuable

tool for numerical modeling, data comparison, and testing alternative hypotheses regarding the nature of dark matter.

In the future, this methodology can be extended to account for relativistic effects, multi-component structures, and dynamically evolving potentials, positioning it as a promising foundation for further research in cosmology and gravitational astrophysics.

Funding

This work was carried out with partial support from Grant FZ-20200929344, awarded by the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan.

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