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Ring-shaped instability of nonequilibrium models of disk-like self-gravitating systems

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In this work, we study the problem of the formation of a ring-shaped structure based on three nonequilibrium disk composite models with an anisotropic velocity diagram, which we constructed. These models are based on a nonstationary generalization of the well-known equilibrium isotropic disk model of Bisnovatyi-Kogan and Zeldovich. Corresponding nonstationary analogues of the dispersion equation for the ring-shaped perturbation mode of the models are introduced. Based on the results of numerical calculations of these equations, a comparative analysis of the three nonequilibrium disk models is carried out, in particular, with respect to the gravitational instability of the ring-shaped perturbation mode.The mechanisms and initial physical conditions for the formation of ring-shaped structures in disk-like self-gravitating systems at the early stage of their evolution have been identified.

Keywords: Self-gravitating disk, ring-shaped structure, nonstationary model, gravitational instability, instability increment

1 Introduction

According to observations, many disk-like galaxies contain various ring-shaped structures, including purely ring galaxies without spiral arms, as well as more complex combinations such as a nucleus, bar, ring, and then spiral arms (Burns, 2004; Mirtadjieva & Nuritdinov, 2016; Buta, 1995, 2017; Kostyuk, 1979). It is well known that one of the main physical mechanisms behind the formation of such large-scale structures in galaxies and other self-gravitating systems is gravitational instability (Abramyan, 1986; Antonov, 1976; Binney & Tremaine, 1987; Kalnajs, 1972; Fridman & Polyachenko, 1984). Today, it is widely recognized that these large-scale structures form during the nonlinear, nonstationary stage of their evolution. Therefore, understanding their origin and revealing the physical properties of galaxies requires

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constructing nonlinear, nonequilibrium models. These analytically solvable models play a crucial role in studying the problem of large-scale structure formation in galaxies, as they not only allow for a mathematical description of the initial states, but also enable investigation of the mechanisms of gravitational instability by introducing appropriate types of perturbations (Mirtadjieva et al. (2025, 2024); Mirtadjieva & Nuritdinov (2023); Mirtadjieva et al. (2022); Mirtadjieva & Mannapova (2021); Mirtadjieva & Nuritdinov (2012a); Mirtadjieva (2012b,c); Mirtadjieva et al. (2011); Mirtadjieva (2009a); Nuritdinov et al. (2009b, 2008)). Furthermore, based on a theoretical approach, we can explain the origin of all characteristic features and identify which processes led to the observed structure, what the initial conditions of formation were, and which nonlinear mechanisms were triggered during the early stages of evolution-mechanisms that remain hidden in numerical simulations. To determine the appropriate criteria for the formation of specific structures in galaxies, it is necessary to construct various nonstationary models that capture the essential features of the early stage of global evolution in real systems. From the perspective of instability theory, it is important that there are no restrictions on the degree of non-stationarity. Simply put, the more parameters used to describe the physical state of the system, the closer the model is to the actual early stages of its evolution, in other words, the more accurately the model can be compared to an observed object. This is why the problem of developing multiparameter models is particularly relevant today. Based on this, in the present work we construct three nonequilibrium composite models of a self-gravitating disk with an anisotropic velocity diagram, and use them to study the gravitational instability of the ring-shaped perturbation mode. For this purpose, nonstationary analogues of the dispersion equations for the ring-shaped perturbation mode are derived, and based on numerical calculations of these equations, graphs of critical dependencies between the model parameters are constructed.

2 Composite nonequilibrium models of a self-gravitating disk

One of the possible approaches to constructing multiparameter models is the method of linear superposition. For instance, if there exist nonlinear, nonstationary two-parameter models of self-gravitating systems that oscillate radially with an arbitrary amplitude λ and phase density functions, $\Psi_{\mathrm{i}}\left(\overrightarrow{\mathrm{r}}, \overrightarrow{\mathrm{v}}, \lambda, \Omega_{\mathrm{i}}, \mathrm{t}\right), \quad (\mathrm{i}=\overline{1, \mathrm{n}}; \quad 0 \leq|\lambda| \leq 1)$, then provided that their corresponding gravitational potentials and surface densities coincide, the principle of superposition can be applied. It is then straightforward to construct a new model in the phase-space description:

$$\Psi\left(\vec{r}, \vec{v}, \Omega_1, \Omega_2, \lambda, \nu, t\right)=(1-\nu) \Psi_1\left(\vec{r}, \vec{v}, \lambda, \Omega_1, t\right)+\nu \Psi_2\left(\vec{r}, \vec{v}, \lambda, \Omega_2, t\right), \qquad (1)$$

where the superposition parameter ν takes values in the interval $[0,1]$, and Ω_{i} is the rotation parameter of the model. Consequently, the nonlinear model (1) immediately becomes anisotropic if at least one of the components is anisotropic, and, moreover, it is a four-parameter model. In model (1), we used nonlinearly pulsating disk models (Mirtadjieva & Nuritdinov (2012a); Nuritdinov et al. (2009b, 2008)) as Ψ_1 and Ψ_2 in various combinations. As a result, we obtained several composite nonequilibrium models of disk-like self-gravitating systems (DSS).

1) Among the various possibilities, particular interest lies in the intermediate states between isotropic and anisotropic models. Therefore, we first constructed a composite model with nonlinear radial pulsation by superposing isotropic and anisotropic nonequilibrium disk models of Nuritdinov (Nuritdinov et al. (2009b, 2008)):

$$\Psi_1 = \frac{\sigma_0}{2\pi\Pi\sqrt{1-\Omega_1^2}}\left[\frac{1-\Omega_1^2}{\Pi^2}\left(1-\frac{\mathrm{r}^2}{\Pi^2}\right)-\left(\mathrm{v_r}-\mathrm{v_a}\right)^2-\left(\mathrm{v_\perp}-\mathrm{v_b}\right)^2\right]^{-1/2}\chi(\mathrm{R}-\mathrm{r}). \tag{2}$$

$$\Psi_2 = \frac{\sigma_0}{\pi}\left[1+\Omega_2\cdot\left(\mathrm{xv_y}-\mathrm{yv_x}\right)\right]\cdot\chi(\mathrm{D}). \tag{3}$$

2) The second composite model was obtained using equation (3) and the model from Mirtadjieva & Nuritdinov (2012a):

$$\Psi_1 = \frac{2\sigma_0}{3\pi}\left\{1+\frac{r^2}{\Pi^2}+\Pi^2\left[\left(\mathrm{v}_r-\mathrm{v}_a\right)^2+\mathrm{v}_\perp^2\right]-3r^2\mathrm{v}_\perp^2\right\}\chi(D). \tag{4}$$

3) Finally, we constructed the third composite nonlinear model of the DSS, where we used equation (3) again for Ψ_2, and for Ψ_1, we applied the model from Mirtadjieva & Nuritdinov (2012a):

$$\Psi_1 = \frac{\sigma_0}{\pi}\left(3\mathrm{D}^2+24\mathrm{D}\cdot\mathrm{r}^2\mathrm{v}_\perp^2+8\mathrm{r}^4\mathrm{v}_\perp^4\right)\chi(\mathrm{D}). \tag{5}$$

Let us recall that here $\mathrm{v_r}$ and $\mathrm{v_\perp}$ are the radial and tangential components of the velocity of the "particle" with coordinates $\vec{\mathrm{r}}(\mathrm{x},\mathrm{y})$; σ_0 is the surface density at the center of the disk at $\mathrm{t}=0$, and χ is the Heaviside function.

$$D = \left(1-\mathrm{r}^2/\Pi^2\right)\left(1-\Pi^2\mathrm{v}_\perp^2\right)-\Pi^2\left(\mathrm{v}_r-\mathrm{v}_a\right)^2\geq 0, \tag{6}$$

$$\mathrm{v}_a = -\lambda\sqrt{1-\lambda^2}\frac{\mathrm{r}\sin\psi}{\Pi^2},\quad \mathrm{v}_b = \frac{\Omega_1\mathrm{r}}{\Pi^2}, \tag{7}$$

where the function $\Pi(\psi)=\left(1-\lambda^2\right)^{-1}(1+\lambda\cos\psi)$ represents the system's pulsation coefficient, and the argument ψ is related to time t in the following way:

$$\mathrm{t} = \left(1-\lambda^2\right)^{-3/2}(\psi+\lambda\sin\psi). \tag{8}$$

Here, the normalization $\pi^2\mathrm{G}\sigma_0 = 2\mathrm{R}_0$, is adopted, where G is the gravitational constant, and R_0 is the radius of the system in its nominal state. For convenience, we assumed that $\mathrm{R}_0 = 1$. Thus, the nonlinear composite models with an anisotropic velocity diagram that we have obtained undergo strict radial oscillations with a period:

$$\mathrm{P}(\lambda) = \frac{2\pi}{\left(1-\lambda^2\right)^{3/2}}, \tag{9}$$

and the corresponding surface density is given by:

$$\sigma(\vec{\mathrm{r}},t) = \frac{\sigma_0}{\Pi^2}\sqrt{1-\frac{r^2}{\Pi^2}}. \tag{10}$$

It is also worth noting that these composite models allow us to investigate intermediate states between two different models, covering a broader range of possible initial conditions at the early stage of DSS evolution, as well as complex states in terms of the anisotropy of the velocity diagram.

3 Study of the instability of the ring-shaped mode of perturbations in the models

To compare these composite models with each other, we studied their instabilities with respect to the ring-shaped mode of perturbation $\mathrm{N} = 4, \mathrm{m} = 0$, which is responsible for the formation of ring structures in galaxies. Recall that N is the primary perturbation index and m is the azimuthal wave number. To investigate the gravitational instability of this mode in the context of these three composite models, we derived the corresponding nonstationary analogues of the dispersion equations (NADE) based on the method described in Mirtadjieva (2012c). Due to their complexity, we will present the NADE for the first composite model here:

$$\Lambda\ell_\tau(\psi) = \frac{45}{8}\left[v\cdot\left(D^2_{04}(\psi) - D^1_{04}(\psi)\right) + D^1_{04}(\psi)\right]\cdot(\lambda+\cos\psi)^{3-\tau}\sin^\tau\psi, \quad \tau = \overline{0-3}. \tag{11}$$

In (11), the function

$$\begin{aligned} D^1_{04}(\psi) = {} & \left[h_1^3\cos^3\psi + h_2\cos^2\psi\sin\psi - h_3\cos\psi\sin^2\psi + h_4\sin^3\psi\right]\ell_0(\psi) + \\ & + \left[3h_1^3\cos^2\psi\sin\psi - h_2\left(q_1\cos^2\psi - 2\cos\psi\sin^2\psi\right) - h_3\left(\sin^3\psi - q_1\sin^2\psi\right) - \right. \\ & \left. -3h_4q_1\sin^2\psi\right]\ell_1(\psi) + \left[3h_1^3\cos\psi\sin^2\psi - h_2\left(q_1\sin^2\psi - \sin^3\psi\right) - \right. \\ & \left. -h_3\left(q_1^2\cos\psi - 2q_1\sin^2\psi\right) + 3h_4q_1^2\sin\psi\right]\ell_2(\psi) + \\ & + \left[h_1^3\sin^3\psi - h_2q_1\sin^2\psi - h_3q_1^2\sin\psi - h_4q_1^3\right]\ell_3(\psi) \end{aligned}$$

and the function

$$\begin{aligned} D^2_{04}(\psi) = {} & \frac{1}{4}h_1^6\left[q_1\left(4q_1^2 - 3q_3\right)\ell_0(\psi) + 3q_2\sin\psi\left(6q_1^2 - q_3\right)\ell_1(\psi) + \right. \\ & \left. +3q_2q_1\left(6q_3 - q_1^2\right)\ell_2(\psi) + q_2^2\sin\psi\left(4q_3 - 3q_1^2\right)\ell_3(\psi)\right], \end{aligned}$$

here

$$\mathrm{h}_1 = (1+\lambda\cos\psi)^{-1}, \quad \mathrm{h}_2 = 3\mathrm{c}\sqrt{\mathrm{q}_2}\mathrm{h}_1^4, \quad \mathrm{h}_3 = \frac{1}{3}\left(4 - 9\mathrm{c}^2 - 7\Omega^2\right)\mathrm{q}_2\cdot\mathrm{h}_1^5$$

$$\mathrm{h}_4 = \left(\mathrm{c}^3 - \frac{4}{3}\mathrm{c} + \frac{7}{3}\mathrm{c}\Omega^2\right)\cdot\mathrm{q}_2^{3/2}\cdot\ \mathrm{h}_1^6, \quad \mathrm{q}_1 = \lambda + \cos\psi, \quad \mathrm{q}_2 = 1-\lambda^2, \mathrm{q}_3 = \mathrm{q}_2\sin^2\psi.$$

Thus, the derived NADE for the ring-shaped mode of perturbations within the framework of the three models were numerically investigated by us using the method of stability of periodic solutions (Malkin (1952)). During the calculations, by varying the values of the parameters Ω, λ, and ν in the range from 0 to 1 , we found the critical values of the initial virial ratio $(2T/\mid \mathrm{U}\mid)_0$, beyond which the composite models become unstable with respect to the ring-shaped perturbation mode (4;0).

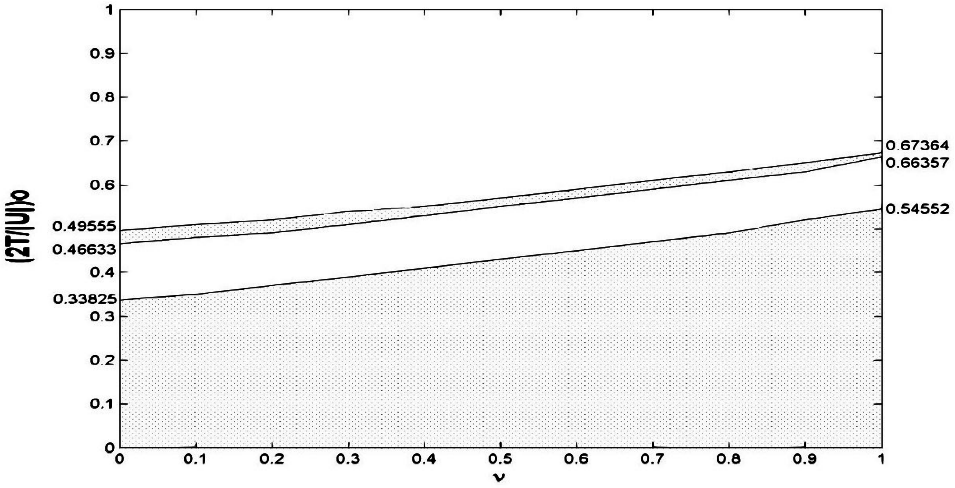


Figure 1 The critical dependence of the initial virial ratio on the superposition parameter in the first composite model for the mode $(4;0)$ at $\Omega_1 = \Omega_2 = 0.4$. (The shaded area corresponds to the instability zone.)

Based on the results of the numerical calculations of the derived NADE, we constructed graphs showing the critical dependencies between the initial virial ratio, the rotation parameter, instability increments, and the superposition parameter of the models. Here is an example of the critical dependence of the initial virial ratio on the superposition parameter for the first composite model at the rotation parameter value $\Omega_1 = \Omega_2 = 0.4$ (Fig. 1).

It should be noted that the stability problem of the isotropic model (2) with respect to the ring-shaped mode $(4, 0)$ was addressed in our work Nuritdinov et al. (2008). The analysis of the obtained results shows that when the rotation parameter $\Omega = 0.4$, the isotropic model (2) is more stable than the anisotropic model (3). In general, considering the superposition parameter ν, we observe a smooth transition from the isotropic to the anisotropic model. Furthermore, the unstable region consists of two zones, separated by a relatively wide zone of stability. For a comparative analysis of the stability of the composite models with respect to this mode, we plotted graphs showing the dependence of the instability increments on the initial virial ratio of the models for various values of the superposition parameter ν, at a rotation parameter value of $\Omega = 0$ (Fig. 2). As can be seen, in the absence of disk rotation, the third composite model consistently dominates in terms of instability. The other two models alternate with each other as the value of the superposition parameter ν changes.It is also worth noting that at $\nu = 0$, the ring-shaped structure forms across the entire range of initial virial ratio values in the context of the first and third models.

However, as the value of the superposition parameter ν increases, the range for the formation of this structure along the $(2\mathrm{T}/|\mathrm{U}|)_0$ axis gradually narrows to the left. Figure 3 presents graphs comparing the instability increments of the ring-shaped mode as a function of the initial virial ratio for the composite models at various values of the superposition parameter ν, with the rotation parameter $\Omega = 0.6$. Here again,

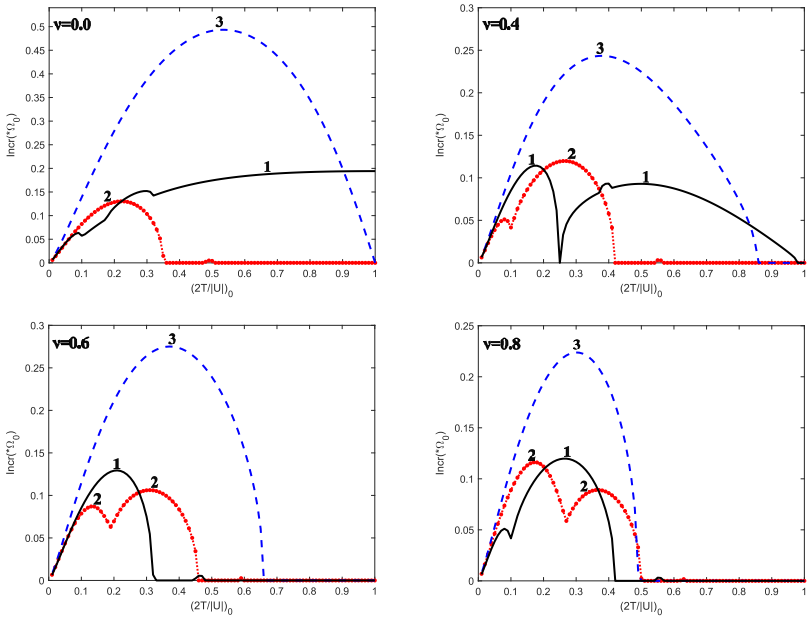


Figure 2 Comparison of instability increments as a function of the initial virial ratio for nonrotating composite models at different values of the superposition parameter ν.

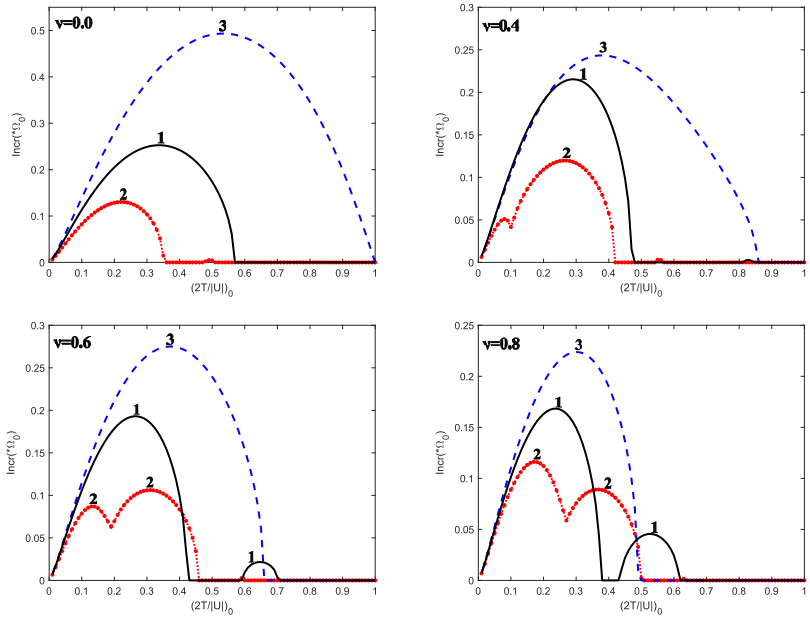


Figure 3 Comparison of instability increments as a function of the initial virial ratio of composite models for different values of the superposition parameter ν, with the rotation parameter $\Omega = 0.6$.

it is evident that the third composite model remains the leading one, and as the superposition parameter increases, the range of initial virial ratio values where the ring-shaped structure forms becomes narrower. More importantly, regardless of the value of the superposition parameter, a consistent pattern emerges in the temporal manifestation of the ring-shaped structure across the models. Specifically, this structure first appears in the third composite model, then in the first, and only afterward in the second. Through analysis of the obtained results, it was revealed that the nonequilibrium composite models exhibit alternating regions of instability of the ring-shaped mode either periodic or aperiodic in nature depending on the values of their parameters.

4 Conclusion

Here we briefly summarize the main results obtained:

- We have constructed three nonequilibrium composite models of a self-gravitating disk with an anisotropic velocity diagram, which make it possible to study intermediate states between two different models, covering wider possible initial conditions at the early stage of the evolution of the DSS and complex states from the point of view of the anisotropy of the velocity diagram.
- The NADE of the ring-shaped perturbation mode have been derived within the framework of the non-equilibrium composite models of a self-gravitat- ing disk.
- Based on numerical solutions of the NADE, graphs of critical dependencies between the model parameters were constructed.
- It was revealed that the nonequilibrium composite models exhibit alternating regions of instability for the ring-shaped mode, which can be either periodic or aperiodic in nature depending on the values of their parameters.
- A comparative analysis of the nonequilibrium composite models with respect to gravitational instability of the ring-shaped perturbation mode shows that, regardless of parameter values, the rate of ring structure formation is always highest in the third model compared to the others.
- It has been shown that as the value of the superposition parameter ν increases, the range of initial virial ratio values within which a ring structure forms becomes narrower, regardless of the models' rotation parameter values.

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