Astronomical and Astrophysical Transactions (*AApTr*), 2025, Vol. 35, Issue 2, pp. 129–134,
ISSN 1055-6796, Photocopying permitted by license only, ©Cambridge Scientific Publishers

Pulsating models of spherical self-gravitating systems

S.N. Nuritdinov*

National University of Uzbekistan named after Mirzo Ulugbek, 100174, Tashkent, Uzbekistan

Received 27 April, 2025; accepted 27 May, 2025

Since gravitational instabilities in reality arise during the non-stationary stage of the evolution of self-gravitating systems, we present below a method for constructing radially pulsating phase models of such systems. Our models pulsate self-similarly, preserving the property of homogeneous volume density. In this work, we provide the corresponding generalizations of two well-known homogeneous equilibrium models: the Einstein model and the Camm sphere. We also indicate the differences in stellar orbits in the pulsating state.

Keywords: Self gravitating, nonstationary model, gravitational instability

1 Introduction

Today it is extremely important to study the early stages of the evolution of galaxies located at distances of billions of light-years. We clearly observe many traces of the early global nonstationarity of galaxies. For example, the bends of the optical and gas disks of S and separate SO galaxies, the regularity of the density profile of E-galaxies, well explained by the mechanism of collisionless relaxation, the presence of powerful corona, anomalous manifestations in the form of ringshaped (with or without a central core) and Υ-shaped shapes, the alternating activity of quasars and radio galaxies etc. All these phenomena, despite their different nature, are the result of global nonstationary processes with significant nonlinearity. However, until now, the theorists of the physics of galaxies solve mainly the equations of evolution by linearizing them near the equilibrium state, that is, considering nonstationarity as a small correction for the stationary model. That is why there is a need to create an attempt at an analytical theory of some of these phenomena in order to identify patterns and identify nonlinear effects that cannot be detected in any way by direct numerical modeling. In recent decades, the problems of the nonstationary stage of the evolution of collisionless self-gravitating systems and the formation of a large-scale structure of galaxies have been

*Email: nur200848@mail.ru

very intensively solved with the help of computer numerical experiments, setting at the initial moment a different degree of non-stationarity. However, during numerical experiments behind the "scene", fundamental questions often remain unnoticed and not clarified, for example: which nonlinear effects and phenomena "worked", whether instability took place, what is the dependence of the evolution of the physical parameters of a non-stationary system on the degree of nonlinearity, and others. It is clear that the solution of such issues is not possible without an analytical approach to such essentially non-linear problems. In this paper, we are primarily interested in the following questions: is it possible to construct analytically solvable models of collapsing galaxies; how to study the problem of studying the stability of such nonlinearly unsteady configurations in order to identify new types of instability against the background of collapse, and not near the same; what are the initial conditions necessary at the moment of the collapse to form different types of galaxies, etc. The direct construction of the models of this or that non-equilibrium state that we need by solving a system of differential Boltzmann–Jeans and Poisson equations is complicated primarily by the mathematical difficulty of the problem, where it is necessary to take into account instability and nonlinearity at the same time. Therefore, without fully considering the nonstationary evolution of a nonlinear nonequilibrium system, we identify typical, realizable states and, accordingly, build analytically accurate nonlinear models of them in the phase description. One of them at the early stage of the evolution of galaxies is the collapsing, gravitationally shrinking or non-linearly pulsating state of the system as a whole.

2 Nonlinearly pulsating generalization of Einstein's equilibrium ball

Let us take as a basis the stationary model of Einstein's ball (Einstein, 1967) with radius R_0, where all particles move only in circular orbits. Obviously, they are kept in circular orbits if the exact equality of gravitational attraction and centrifugal force is satisfied. In this case, the value of the velocity vector $\overrightarrow{v_0}$ in the equilibrium state is equal to

$$v_0 = \Omega_0 r_0, \quad \Omega_0 = \left[(4/3) \cdot \pi G \rho_0\right]^{1/2}, \tag{1}$$

where Ω_0 is the angular velocity of rotation of the star, ρ_0 is the volume density. To build a nonlinear model, it is necessary to have a configuration that is completely stable in the linear approximation. Therefore, we consider the model in the form of a homogeneous Einstein ball, since in an inhomogeneous system, under radial perturbations, generally speaking, intersections of layers can occur, which leads to instability already in the case of linear oscillation amplitude. Einstein's stationary model is extremely anisotropic in terms of velocities, since in it the radial component of the energy of motion is strictly zero. This model is stable with respect to small asymmetric perturbations (Fridman, 1984). But it is clear that when the peculiar velocities are included in it, various instabilities may occur. To find out at what values this is possible, we first need to build a non-linearly pulsating model. For the first time (Antonov, 1981) found a pulsating state of Einstein equilibrium nonrotating

model considering disturbed coordinates

$$\overrightarrow{\mathrm{r}} = \alpha(\mathrm{t}) \cdot \overrightarrow{\mathrm{r}_0} + \beta(\mathrm{t}) \cdot \overrightarrow{\mathrm{v}}_0, \tag{2}$$

$$\overrightarrow{\mathrm{v}} = \dot{\alpha}(t) \cdot \overrightarrow{r_0} + \dot{\beta}(t) \cdot \overrightarrow{\mathrm{v}_0}, \tag{3}$$

where $\overrightarrow{r_0}$ and $\overrightarrow{\mathrm{v}_0}$ are, respectively, the radius vector and the velocity vector in the main stationary state. Since there are no radial motions in the Einstein ball in the initial state, $\overrightarrow{r}_0 \cdot \overrightarrow{v}_0 = 0$, we easily find

$$\mathrm{r} = \Pi(\mathrm{t}) \cdot \mathrm{r}_0, \quad \Pi(\mathrm{t}) = \left(\alpha^2 + \Omega_0^2 \beta^2\right)^{1/2}. \tag{4}$$

And also we obtain the following solution

$$\Pi(\psi) = \left(\Omega_0^2/\mathrm{q}^2\right)(1 + \lambda \cos\psi), \quad \lambda = \left[1 - \left(\mathrm{q}^2/\Omega_0^2\right)\right]^{1/2}, \tag{5}$$

which satisfies the condition $t = 0, \dot{\Pi} = 0$ (that corresponds to the moment of greatest expansion). Thus, we have found the expansion coefficient Π of the Einstein ball during its pulsation. The value λ plays the role of the pulsation amplitude and takes values in the interval [0,1]. According to (2) and (3), we have $\overrightarrow{\mathrm{r}} \times \overrightarrow{\mathrm{v}} = \overrightarrow{r_0} \times \overrightarrow{\mathrm{v}_0}$. This implies, what $\mathrm{rv}_\perp = \mathrm{r}_0 \mathrm{v}_{\perp 0} = \Omega_0 \mathrm{r}_0^2$ or

$$\mathrm{v}_\perp = \Omega_0 \mathrm{r}_0 / \Pi(\psi) = \Omega_0 \mathrm{r} / \Pi^2(\psi) \equiv \mathrm{v}_\mathrm{b}. \tag{6}$$

Since $\overrightarrow{\mathrm{r}} \cdot \overrightarrow{\mathrm{v}} = \left(\alpha\dot{\alpha} + \Omega_0^2 \beta\dot{\beta}\right) \mathrm{r}_0^2 = \mathrm{rv}_\mathrm{r}$, taking into account (1.13) and (1.26), we have

$$\mathrm{v}_\mathrm{r} = -\Omega_0 \lambda \left(1 - \lambda^2\right)^{-3/2} \sin\psi \cdot (1 + \lambda\cos\psi)^{-2} \mathrm{r} \equiv \mathrm{v}_\mathrm{a}. \tag{7}$$

Now we can write an expression for the phase density of a nonlinearly pulsating version of the Einstein model in the form

$$\Psi\left(\mathrm{r}, \mathrm{v}_\mathrm{r}, \mathrm{v}_\perp, \mathrm{t}\right) = \rho(\mathrm{t}) \left[2\pi \mathrm{v}_\mathrm{b}(\mathrm{t})\right]^{-1} \delta\left(\mathrm{v}_\mathrm{r} - \mathrm{v}_\mathrm{a}\right) \delta\left(\mathrm{v}_\perp - \mathrm{v}_\mathrm{b}\right) \chi\left(\mathrm{R}_0 \Pi - \mathrm{r}\right), \tag{8}$$

where χ is the Heaviside function, δ is the Dirac function. The non-stationary model (8) pulsates with a period

$$P(\lambda) = q^{-1} \cdot \int_0^{2\pi} \Pi(\psi) d\psi = (2\pi/\Omega_0) \cdot \left(1 - \lambda^2\right)^{-3/2}. \tag{9}$$

In this case, the individual particles of the system describe, obviously, elliptical orbits with a focus at the center of the system.

3 Pulsating version of equilibrium Camm model

We have a possibility of generalizing to the case of nonlinear pulsations for the well-known equilibrium Camm model with the phase density

$$\Psi_0 = \mathrm{c} \cdot \left[\Omega_0^2 \cdot \left(\mathrm{R}_0^2 - \mathrm{r}_0^2\right) - \mathrm{v}_{0\perp}^2 \cdot \left(1 - \mathrm{r}_0^2/\mathrm{R}_0^2\right) - \mathrm{v}_{0\mathrm{r}}^2\right]^{-1/2} \cdot \chi\left(\mathrm{R}_0 - \mathrm{r}_0\right), \tag{10}$$

where c is a constant, $\mathrm{v_{0r}}$ and $\mathrm{v_{0\perp}}$ are the radial and transversal components of the velocity vector $\mathrm{v_0}$. In contrast to the Einstein equilibrium ball, model (10) contains stellar orbits with arbitrary ellipticity, including circular ones, and the center of the system coincides with the center of these orbits. However, model (10), just like Einstein's equilibrium ball, is homogeneous and has a quadratic gravitational potential.

For the considered fluctuations, the quadratic property of the potential is preserved. Consequently, the equation of motion of an individual star in a state of pulsation has a known form. This means that model (10) can be generalized to the nonstationary case with the pulsation law (4). To this end, we substitute $\overrightarrow{r_0}$ and $\overrightarrow{v_0}$ from (2) and (3) into the equation of the phase boundary of the original model (10)

$$\mathrm{v_0^2} + \Omega_0^2 \cdot \mathrm{r_0^2} - \left(\overrightarrow{\mathrm{r_0}} \times \overrightarrow{\mathrm{v_0}}\right)^2 / \mathrm{R_0^2} = \Omega_0^2 \cdot \mathrm{R_0^2}\,. \tag{11}$$

Then we have

$$\left(\dot{\alpha}^2 + \Omega_0^2 \dot{\beta}^2\right) \mathrm{r}^2 - 2\left(\alpha\dot{\alpha} + \Omega_0^2 \beta\dot{\beta}\right) \overrightarrow{\mathrm{r}} \cdot \overrightarrow{\mathrm{v}} + \Pi^2(\mathrm{t})\mathrm{v}^2 - \left(\overrightarrow{\mathrm{r}} \times \overrightarrow{\mathrm{v}}\right) \cdot \mathrm{R_0^{-2}} = \Omega_0^2 \cdot \mathrm{R_0^2}\,. \tag{12}$$

In (12) we divide the speed into its components $(\mathrm{v_r}, \mathrm{v_\perp})$ and transform it into the form

$$\Pi^2 \left[\mathrm{v_r} - \left(\alpha\dot{\alpha} + \Omega_0^2 \beta\dot{\beta}\right) \Pi^{-2} r\right]^2 + \left(\Pi^2 - r^2/R_0^2\right) \cdot \left(\mathrm{v_\perp^2} - \Omega_0^2 R_0^2/\Pi^2\right) = 0\,. \tag{13}$$

This shows that the largest admissible value of r, at which

$$\mathrm{v_r} - \left(\alpha\dot{\alpha} + \Omega_0^2 \beta\dot{\beta}\right) \cdot \mathrm{r}/\Pi^2 = 0$$

is

$$R(t) = \max r = R_0 \Pi\,. \tag{14}$$

Taking into account (13), we can easily write the phase density of the pulsating model

$$\Psi\left(\mathrm{r}, \mathrm{v_r}, \mathrm{v_\perp}, \mathrm{t}\right) = \rho\Pi^4 / \left(\pi^2 \mathrm{R}^2 \Omega_0^2\right) \cdot \mathrm{f}^{-1/2} \cdot \chi(\mathrm{f})\,, \tag{15}$$

where

$$\mathrm{f} = \left(1 - \mathrm{r}^2/\mathrm{R}^2\right) \cdot \left(\Omega_0^2 \mathrm{R}^2/\Pi^4 - \mathrm{v_\perp^2}\right) - \left\{\mathrm{v_r} + \Omega_0 \lambda \cdot \sin\varphi \cdot \mathrm{r} / \left[\left(1 - \lambda^2\right)^{1/2} \cdot \Pi^2\right]\right\}^2. \tag{16}$$

Function (15) satisfies the condition $\rho = \int \Psi \mathrm{d}\overrightarrow{\mathrm{v}}$. In model (15), the trajectory of each star, in contrast to (8), is only elliptical (non-circular), if, of course, the energy of radial oscillations is less than the parabolic limit. In this case, the center of the system can be in the focus of an elliptical orbit, if the initial orbit ($\lambda = 0$) is not circular. Indeed, let the particle be in the initial state, for example, in the plane x_0, y_0, i.e. $\overrightarrow{r_0} = \vec{r}_0\,(x_0, y_0, 0)$, and $\overrightarrow{v_0} = \overrightarrow{v_0}\,(0, v_0, 0)$. Then using the relations

$$x = \alpha x_0 = \left(1 - \lambda^2\right)^{-1} \cdot (\lambda + \cos\psi) \cdot x_0, \quad \mathrm{y} = \alpha \mathrm{y_0} + \mathrm{q}^{-1} \sin\psi \cdot \mathrm{v_0}\,,$$

we get the following orbit equation:

$$\left[\left(1 - \lambda^2\right) \cdot x \cdot x_0^{-1} - \lambda\right]^2 + \left(\Omega_0/\mathrm{v_0}\right)^2 \cdot \left(1 - \lambda^2\right) \cdot \left(y - y_0 x_0^{-1} x\right)^2 = 1. \tag{17}$$

This shows that at $\lambda \neq 0$ the center of the orbit is displaced relative to the origin of coordinates. Further, passing in (17) from (x,y) to polar coordinates (r, φ), the quadratic equation for r, whose discriminant is equal to

$$\left(\Omega_0^2 y_0^2/v_0^2 + 1\right) \cdot \cos^2 \varphi/x_0^2 - \Omega_0^2 y_0 \cdot \sin 2\varphi/\left(x_0 v_0^2\right) + \Omega_0^2 \cdot \sin^2 \varphi/v_0^2 .$$

In the general case, it is impossible to extract the root from the last expression to derive the known dependence $r(\varphi)$ in the form of an equation of a conic section. In order to obtain such an equation, in (17) it is necessary to shift the focal point relative to the center of the system. However, this shift does not need to be performed for the pulsating state (8) of the Einstein ball, which in the equilibrium state in the plane (x_0, y_0) contains only a circular orbit: $v_0 = \Omega_0 \cdot x_0$ and $y_0 = 0$. Finally, it should be emphasized that in the future it is necessary to take into account the presence of a corona of galaxies. This is important for the subsequent stages of their nonstationary evolution. Unfortunately, it has not yet been possible to analytically construct a nonlinear nonstationary model of a collapsing galaxy with an exact nonequilibrium law in the presence of a corona.

4 Conclusion

In this work, we presented a method for constructing pulsating phase models under the condition of preserving the property of homogeneous volume density. These pulsating models are primarily necessary for studying the mechanisms of gravitational instabilities that arise against the background of the system's pulsation. We constructed pulsating states for the well-known equilibrium models of Einstein and Camm. These generalizations were carried out for non-rotating models, and to obtain rotating models, it is sufficient to multiply the phase density by a known expression involving the integral of areas.

References

Antonov, V.A. (1981). Dynamical Stability of Systems of Gravitating Masses. Soviet Astronomy, 25, 123–128.

Einstein, A. (1967). *Relativity: The Special and General Theory.* Crown Publishers.

Fridman, A.M. & Milyukov, A.A. (1984). *Physics of Gravitating Systems.* Nauka Publishing.