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Construction of a probabilistic cloud of trajectories under strong nonlinearity in the orbit determination problem

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The paper uses a method for detecting two lines of variation in the initial confidence domain with strong nonlinearity in the problem of least-square orbit determination. The method is based on the use of a curved coordinate system defined by two lines of variation and the remaining four rectilinear axes. In this coordinate system, the initial confidence region is a six-dimensional unit sphere. The sphere is filled with random points according to the law of Gaussian multidimensional distribution, which gives the initial ensemble of trajectories in these coordinates. The method was tested on constructing the initial cloud of trajectories of three asteroids with short observational arcs.

Keywords: LOV, confidence domain, nonlinearity, orbit determination

1 Introduction

The work continues the author's research (2024) related to the determination of lines of variation (LOV) in the initial confidence region and the construction of a probabilistic trajectory cloud based on them. A number of works are devoted to these issues, among which one can mention (Milani, 1999; Del Vigno, 2020; Letner et al., 2022).

"Lines of variation" ("LOV") refers to the lines of greatest elongation of the initial confidence area. When the nonlinearity in the orbit determination problem is weak, these lines practically coincide with the axes of the confidence ellipsoid. The problem of determining orbits has a large gully, so the confidence ellipsoid is noticeably elongated, i.e. its semi-axes usually differ by several orders of magnitude. The author's work (2024) shows that with strong nonlinearity, the confidence ellipsoid deforms only along its two largest axes, while the other four axes remain practically unchanged. Therefore, it is sufficient to determine only two lines of variation (let us call them LOV1 and LOV2) corresponding to the two largest axes of the ellipsoid.

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Table 1 RMS errors at the ends of the semi-axes of the confidence ellipsoid, the nonlinearity coefficients χ_1, χ_2 and the values of σ_0, b.

Asteroid's designation:	2020 RS179	2017 SU32	2017 SM16
Observations, arc:	17 obs., 3 days	38 obs., 1.5 days	29 obs., 6 days
Semi-axis number		RMS errors	
1	$1.797910''$	$0.3427772''$	$0.6268303''$
2	$1.215144''$	$0.3283370''$	$0.5827966''$
3	$0.3511933''$	$0.2528901''$	$0.4725044''$
4	$0.3512568''$	$0.2528672''$	$0.4725200''$
5	$0.3511891''$	$0.2528282''$	$0.4725104''$
6	$0.3511891''$	$0.2528282''$	$0.4725102''$
7	$0.3511860''$	$0.2528270''$	$0.4725119''$
8	$0.3511860''$	$0.2528276''$	$0.4725119''$
9	$0.3511891''$	$0.2528274''$	$0.4725107''$
10	$0.3511891''$	$0.2528274''$	$0.4725107''$
11	$0.3511860''$	$0.2528282''$	$0.4725116''$
12	$0.3511860''$	$0.2528282''$	$0.4725116''$
χ_1, χ_2	13.71, 0.00065	2.72, 0.0019	1.92, 0.00019
σ_0	$0.2973949''$	$0.2363145''$	$0.4322960''$
b	0.001579351	0.00039889067	0.001796908

In this paper, the further improvement of the method of constructing an ensemble of trajectories described in the author's work (2024) is carried out. The method described in the author's paper (2024) is used to detect the lines of variation. It is based on the search for the conditional maximum of the modulus of the normal vector to the level surface of the objective function of the least squares method. The RMS error value at the ends of the smallest axis of the confidence ellipsoid is used as a condition defining a separate level surface. At the ends of the smallest axis of the confidence ellipsoid, the level surface of the objective function of the least squares method coincides with the surface of the ellipsoid. This makes it possible to use these values of the RMS error as constants specifying the corresponding level surfaces.

As shown in the author's works (2023, 2024), the square of the RMS error σ at the ends of the smallest axis of the ellipsoid depends quadratically on the magnification coefficient k of the ellipsoid: $\sigma^2 = \sigma_0^2 + bk^2$, where σ_0 is the RMS error in the center of the ellipsoid; b is a constant value. These values σ are used as constants that define the level surfaces corresponding to a given magnification factor k.

Three asteroids with a short observational arc were used as model objects. For them, the orbit was determined by the least squares method, and the standard errors at the ends of all 12 semi-axes of the confidence ellipsoid were calculated at k from 0.1 to 6 in increments of 0.1. For $k = 4.5$, the coefficients of nonlinearity χ_1, χ_2 are calculated, defined as (Chernitsov et al., 2014) $0.5(\sigma_{max} - \sigma_{min})/(\sigma_{min} - \sigma_0)$, where σ_{max} is the maximum RMS, respectively, at the ends of the first or the second axis; σ_{min} is the minimum RMS at the ends of the 12 semi-axes of the confidence ellipsoid. The calculation results for $k = 4.5$ are shown in Table 1.

The values of b are obtained by approximating the square of the RMS error calculated at the end of the smallest axis of the ellipsoid using the formula $\sigma^2 = \sigma_0^2 + bk^2$.

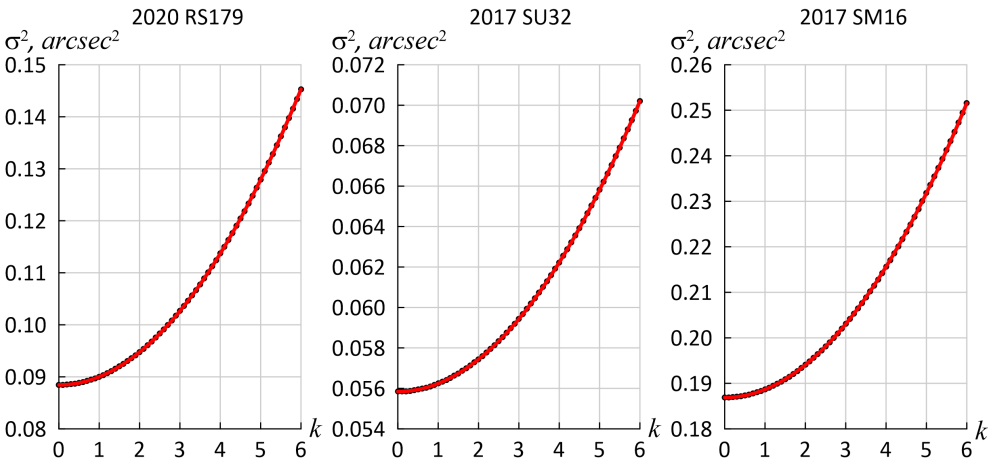


Figure 1 The results of approximating the square of the RMS error calculated at the end of the smallest axis of the confidence ellipsoid.

For this purpose, the values of this RMS error for k were calculated from 0.1 to 6 in increments of 0.1. The results are shown in Fig. 1.

The coefficient of determination R^2 of the value of b in all three cases is greater than 0.9999999999999999544, which means that the dependence $\sigma^2 = \sigma_0^2 + bk^2$ is practically accurate. The coefficient R^2 is determined by the formula $R^2 = 1 - s^2/s_k^2$, where s^2 is the sum of the squares of the differences between the given sigma values and those obtained by the approximating formula; s_k^2 is the sum of the squares of the differences between the given sigma values and their average value. The approximating curve is shown in Fig. 1 in red. (Remark: all calculations in the work are performed using 80-bit floating point numbers, i.e. with 19-digit decimal precision; the GNU-fortran compiler is used to compile programs.)

2 Calculating the lines of variation

Table 1 shows that with strong nonlinearity, the first (largest) axis of the confidence ellipsoid is most deformed and, to a lesser extent, the second axis. The other four axes remain virtually unchanged. Therefore, it is sufficient to obtain only two lines of variation (let us call them LOV1 and LOV2) corresponding to the two directions of the greatest elongation of the confidence region. In the center of the confidence ellipsoid, the directions of the LOV1 and LOV2 coincide with the directions of the two largest axes of the ellipsoid. As they move away from the center of the ellipsoid, LOV1 and LOV2 deviate more and more from these axes.

The method described in the introduction was used to determine LOV1 and LOV2. When setting the level surfaces, σ values were used, calculated using the approximating formula $\sigma^2 = \sigma_0^2 + bk^2$. The $\xi_1, \ldots, \xi_6$ system was used as a working coordinate system, the axes of which coincide with the axes of the confidence ellipsoid, and the

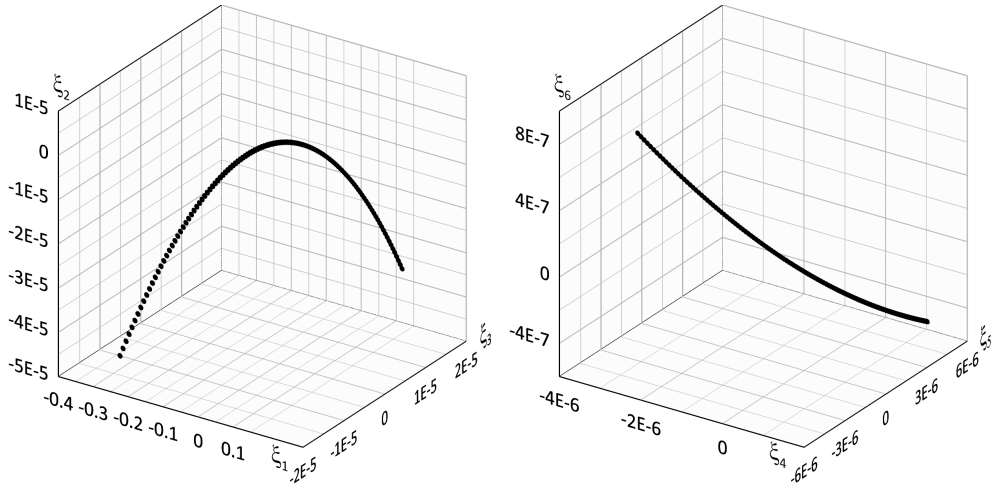


Figure 2 The results of calculating the LOV1 for the asteroid 2020 RS179.

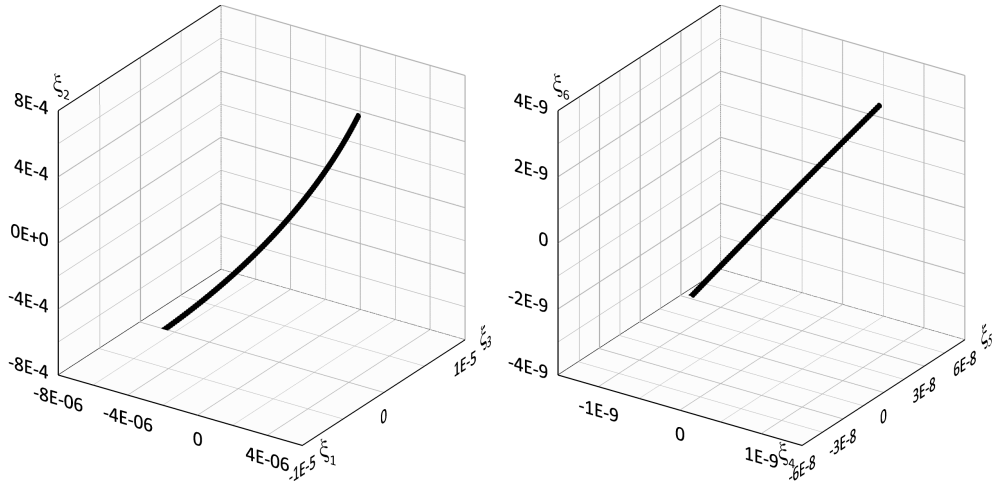


Figure 3 The results of calculating the LOV2 for the asteroid 2020 RS179.

origin of the coordinates coincides with its center. The coefficient k, which determines the points of the lines of variation along the positive direction of the axis, ran values from 0.1 to 6 in increments of 0.1. For the negative direction of the axis, this coefficient was also conditionally assumed to be negative, i.e. it ran values from –0.1 to –6.

The results of calculating the LOV1 and LOV2 are shown in Figs 2–7 as two 3D projections on three axes of the $\xi_1, \ldots, \xi_6$ coordinate system.

To check the accuracy of calculating the points of the lines of variation, we can calculate the probability density f_{non} at these points (we will call this density "nonlinear") by the formula $f_{non} = 1/((2\pi)^3 l_1 \ldots l_6) \exp(-k^2/2)$, where $l_1, \ldots, l_6$ are the

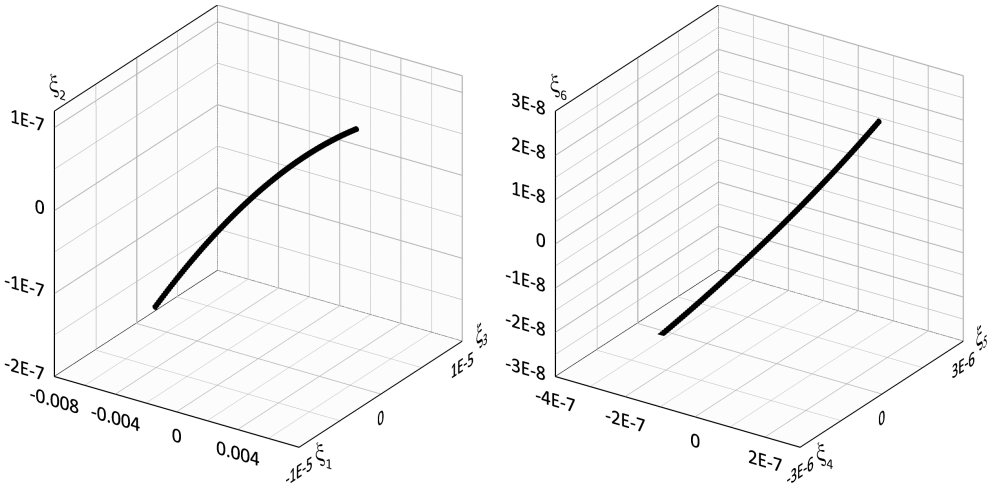


Figure 4 The results of calculating the LOV1 for the asteroid 2017 SU32.

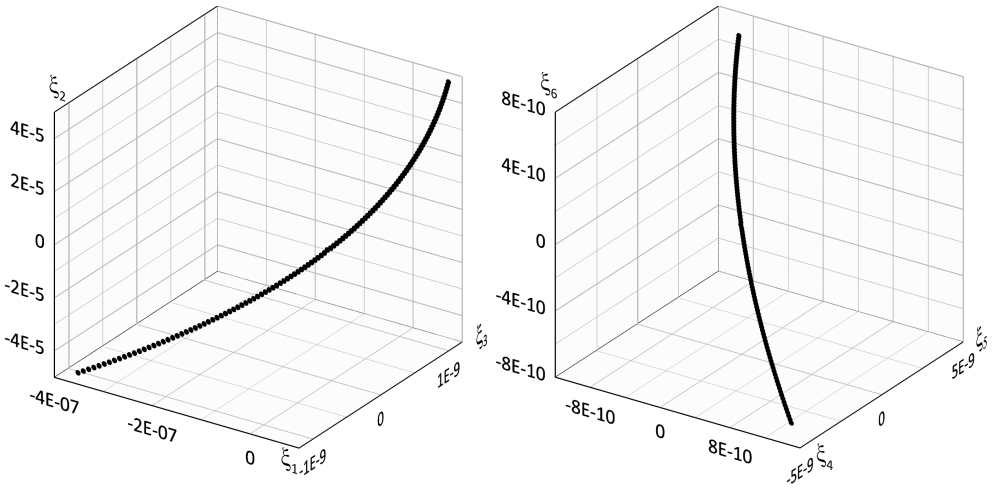


Figure 5 The results of calculating the LOV2 for the asteroid 2017 SU32.

lengths of the semi-axes of the confidence ellipsoid at $k = 1$; $k^2 = (\sigma^2 - \sigma_0^2)/b$ is expressed from the formula $\sigma^2 = \sigma_0^2 + bk^2$ (σ is calculated at a given point of the LOV1 or LOV2).

The theoretical value f of this density is calculated using a similar formula, in which k is the magnification coefficient of the ellipsoid, i.e. $k^2 = (\xi_1/l_1)^2 + \ldots (\xi_6/l_6)^2$ (for the LOV1 $k = \xi_1/l_1$, for the LOV2 $k = \xi_2/l_2$).

Graphs of the relative difference $|\Delta f|/f$ of the densities f_{non} and f are shown in Fig. 8. As can be seen from the graphs, this difference does not exceed 10^{-5}, therefore it can be concluded that the points of the LOV1 and LOV2 are determined quite accurately.

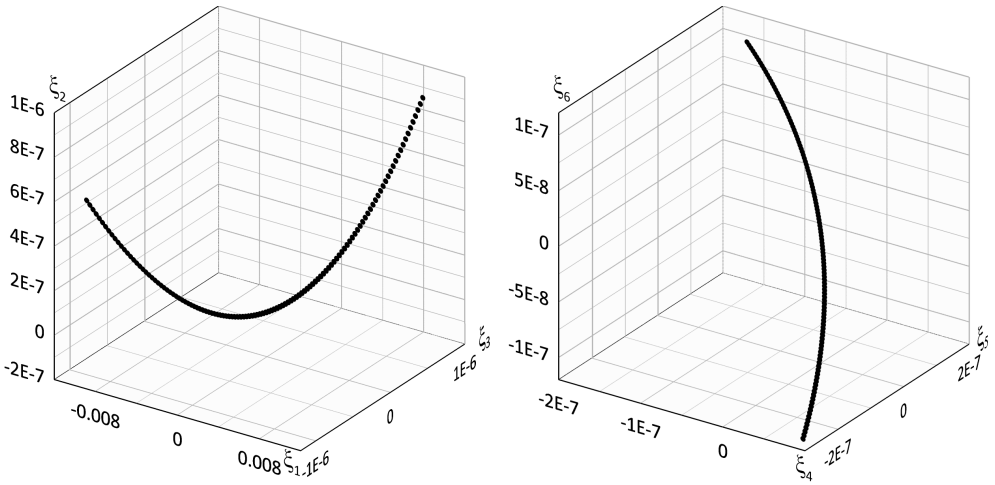


Figure 6 The results of calculating the LOV1 for the asteroid 2017 SM16.

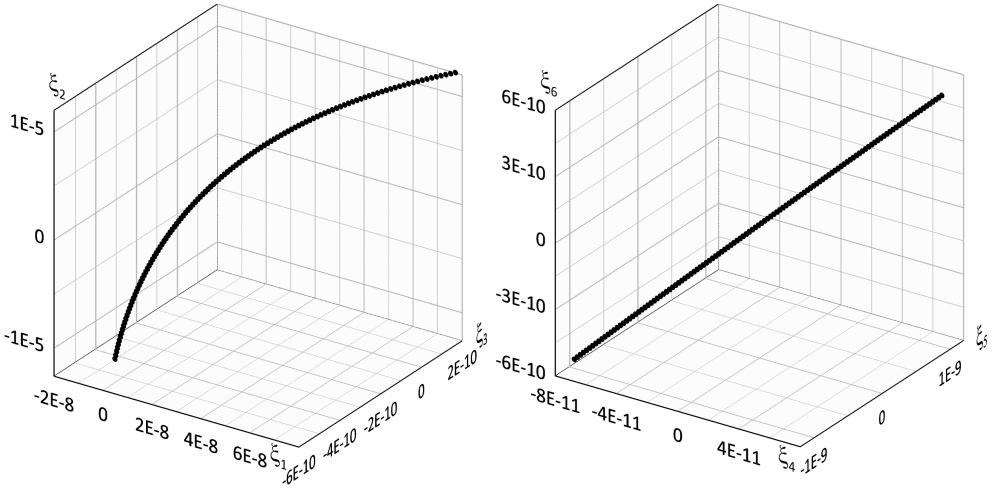


Figure 7 The results of calculating the LOV2 for the asteroid 2017 SM16.

3 Approximation of lines of variation and construction of a trajectory cloud

Since the points LOV1 and LOV2 were calculated for specific values of k, third degree polynomials of the form $u_1k+u_2k^2+u_3k^3$ were used to approximate the lines of variation (since the lines of variation always pass through the origin). For each coordinate $\xi_1, \ldots, \xi_6$, a separate approximating polynomial with an independent parameter k was constructed. The minimum coefficient of determination R^2 for all polynomial coefficients was 0.99994, which indicates a fairly good accuracy of the approximations obtained. Thus, the resulting approximating polynomials can be considered as para-

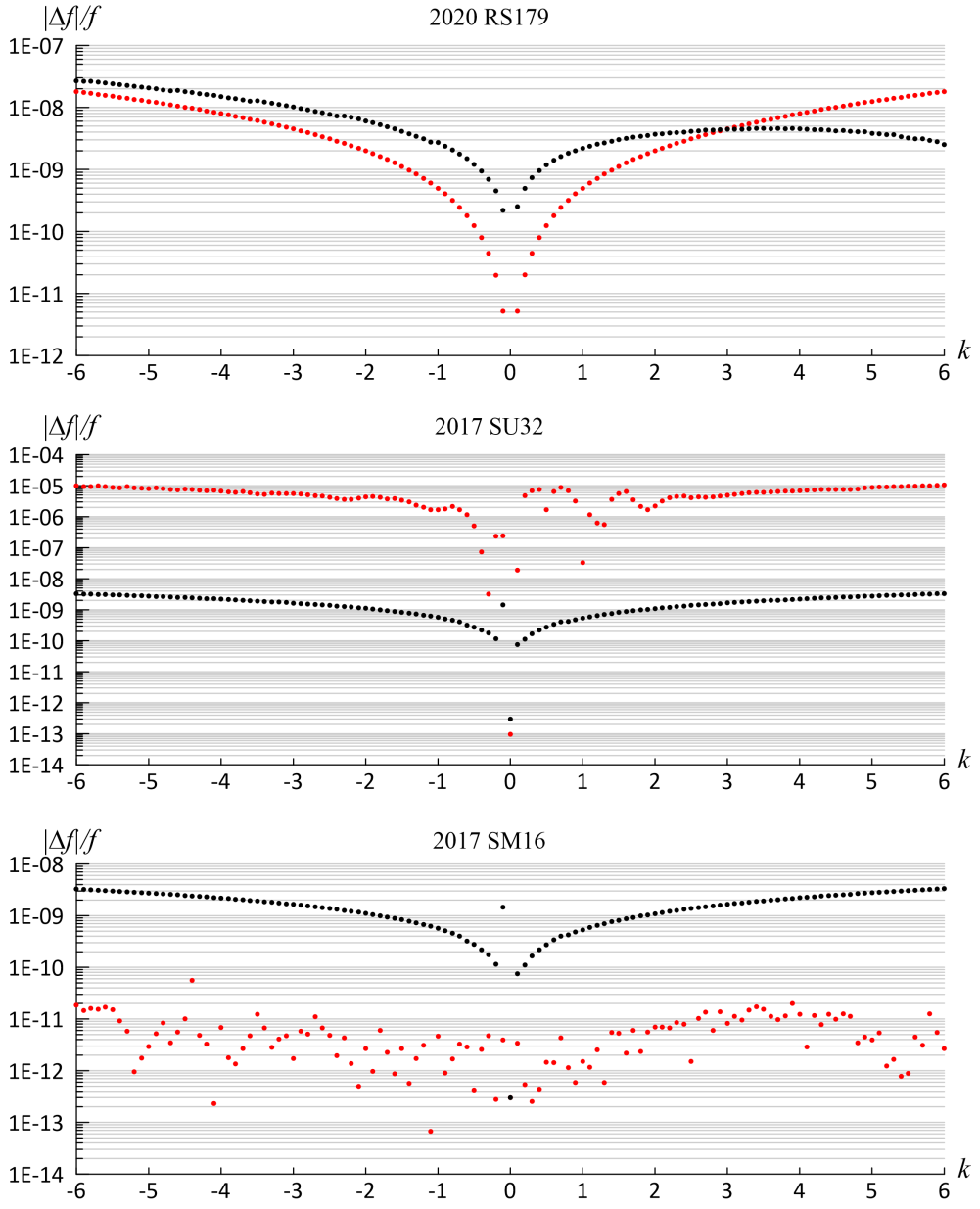


Figure 8 The relative difference of the nonlinear probability density along the LOV1 (black) and LOV2 (red) from the theoretical value.

metric equations of the lines of variation (with two independent parameters k_1 and k_2). Accordingly, for LOV1 and LOV2 they look like this:

$$\xi_1 = u_1(k_1) = u_{11}k_1 + u_{12}k_1^2 + u_{13}k_1^3, \quad \xi_1 = v_1(k_2) = v_{11}k_2 + v_{12}k_2^2 + v_{13}k_2^3,$$
$$\cdots\cdots\cdots \qquad\qquad \cdots\cdots\cdots$$
$$\xi_6 = u_6(k_1) = u_{61}k_1 + u_{62}k_1^2 + u_{63}k_1^3, \quad \xi_6 = v_6(k_2) = v_{61}k_2 + v_{62}k_2^2 + v_{63}k_2^3.$$

Vector functions $\mathbf{u}(k_1)$ and $\mathbf{v}(k_2)$ define the "main" coordinate surface "stretched" on the curved axes LOV1 and LOV2. The remaining four coordinates can be obtained by shifting the point of this surface parallel to the other four axes of the confidence ellipsoid, by entering four other independent parameters $k_3 = \xi_3/l_3, k_4 = \xi_4/l_4, k_5 = \xi_5/l_5$ and $k_6 = \xi_6/l_6$):

$$\begin{aligned}
\xi_1 &= u_1(k_1) + v_1(k_2),\\
\xi_2 &= u_2(k_1) + v_2(k_2),\\
\xi_3 &= u_3(k_1) + v_3(k_2) + hl_3k_3,\\
\xi_4 &= u_4(k_1) + v_4(k_2) + hl_4k_4,\\
\xi_5 &= u_5(k_1) + v_5(k_2) + hl_5k_5,\\
\xi_6 &= u_6(k_1) + v_6(k_2) + hl_6k_6,
\end{aligned} \tag{1}$$

where h is a correction factor that allows us to slightly "tighten" the points to a level surface with the desired probability density. This density is determined by the formula $f = 1/((2\pi)^3 l_1 \dots l_6)\exp(-k^2/2)$, where $k^2 = k_1^2 + \dots k_6^2$.

The method of "pulling up" to the desired level surface is the following. Let us denote by $\mathbf{s}(h)$ the vector of left-hand sides of (1) at fixed values $k_1, \dots, k_6$, i.e. $\mathbf{s}(h) = (\xi_1, \xi_2, \xi_3(h) \dots, \xi_6(h))^T$. To find the required value of h, the following equation is solved:

$$\sigma^2(\mathbf{s}(h)) - \sigma_0^2 = bk^2. \tag{2}$$

Equation (2) is solved with respect to h by the iterative method of differential corrections. After that, the resulting solution h is supplied to (1) and a point corresponding to the given vector $\mathbf{k} = (k_1, \dots, k_6)^T$ is obtained.

As for the vectors $\mathbf{k}$, they are set when constructing a trajectory cloud in accordance to a 6-dimensional Gaussian distribution with unit covariance matrix. The algorithm for obtaining such a spread of points is described, for example, in (Pollak, 1971).

The described method was applied to build a trajectory cloud for the three asteroids used. According to the described algorithm, 10,000 points were calculated, for each of which a relative difference $|\Delta f|/f$ in probability density was calculated (as before for the lines of variation LOV1 and LOV2). The results are shown in Figs 9–11. The lower graphs show the values of the correction factor h for each point.

As can be seen from Figs 9–11, the relative difference between the probability density and its theoretical value does not exceed 10^{-9}, which means that the nonlinear density is practically equal to its theoretical value. The values of the correction factor h are located close to one, which means that only a small shift along the coordinate axes is enough to achieve the desired density.

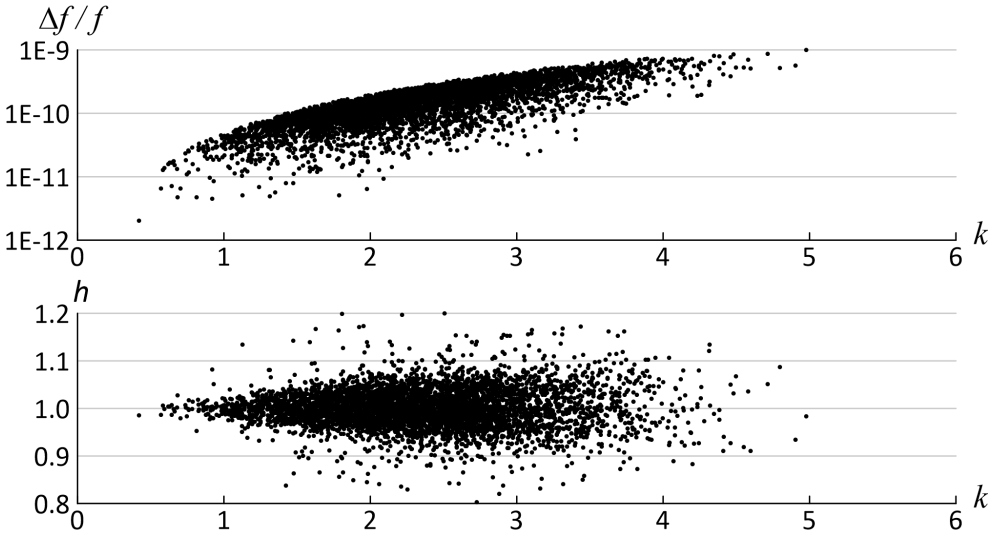


Figure 9 The relative difference in probability density and the correcting factor h for the cloud points of the asteroid 2020 RS179.

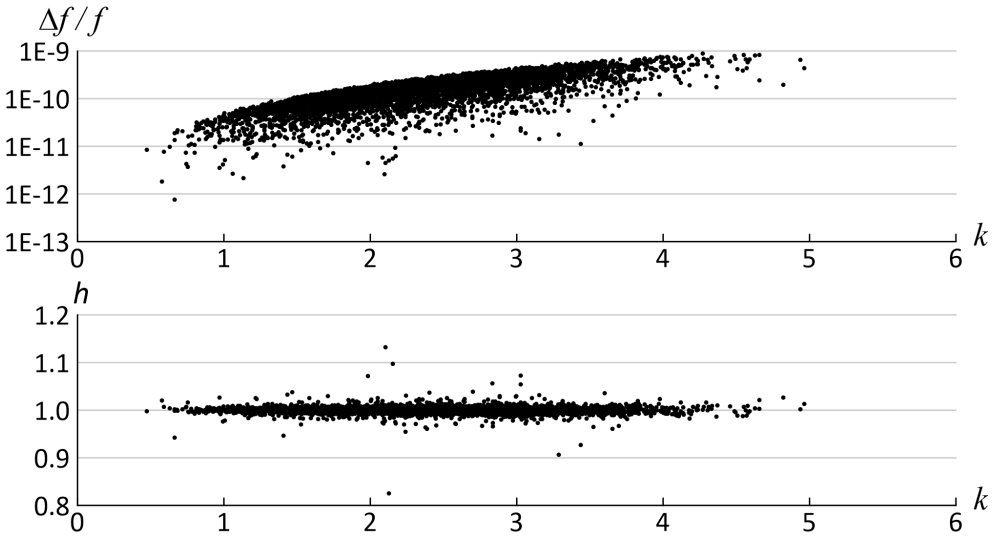


Figure 10 The relative difference in probability density and the correcting factor h for the cloud points of the asteroid 2020 SU32.

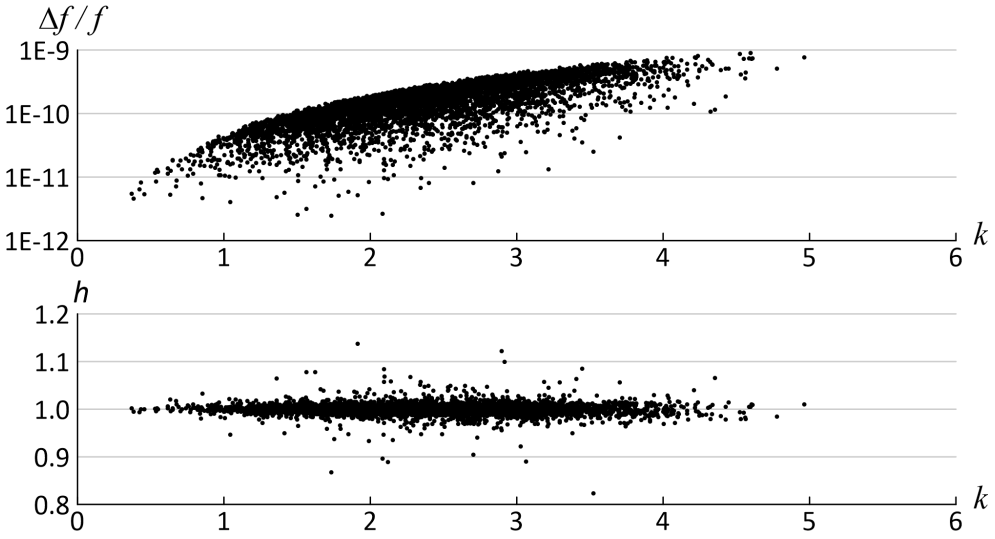


Figure 11 The relative difference in probability density and the correcting factor h for the cloud points of the asteroid 2017 SM16.

4 Conclusions

In the work, a method was used to determine the lines of variation in the initial confidence domain, based on the search for the conditional maximum of the square of the modulus of the normal vector to the level surfaces of the minimized function of the least squares method. It is shown that the method allows us to determine the points of LOV1 and LOV2 quite accurately, since the density of the confidence probability is almost equal to the density at the corresponding points of the axes of the confidence ellipsoid.

New variables were used to define the confidence domain, in which the domain is a unit sphere. A method has been developed for mapping points of such a domain into real phase space using cubic approximations of LOV1 and LOV2. Testing of the method showed that at the obtained points of the trajectory cloud, the probability density is almost equal to its theoretical value.

The advantage of the developed method is that asteroid observations are constants and do not vary. This corresponds to the nature of observations, unlike, for example, the method of perturbed observations, which is quite often used in the construction of confidence areas.

The method is a further improvement of the method described in the author's work (2024). The improved method makes it possible to obtain a more accurate confidence domain with strong nonlinearity in the problem of determining the orbit.

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