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Towards multiscale analysis of the structure of anisotropic conglomerations: from micro-mechanical (granular) to Universe

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We show that by applying the Dirichlet problem together with Mises-transform approach in the context of multi-level structural analysis it is possible to relate surface characteristics to volumetric structure in various physical systems, with accuracy controlled. The typical examples are the multilevel large-scale structure of the Universe and the structure of granular materials on a mesoscale. By sequentially solving relevant boundary value problems for structural elements at each scale, one can estimate for instance, and relate mass distribution on different levels (cross-sections) of observations. In this case, characteristics obtained at a smaller scale (low-dimensional cross-sections) can be used as input data at a larger level. The Dirichlet problem is applied to describe the boundary between regions with different concentrations of matter or different structures.

Keywords: Multiscale analysis, anisotropic matter, granular materials, Universe, Dirichlet problem, Mises transform, structure of matter

1 Introduction

The problem of recovering 3D information from 2D images is common in many areas of science. For example, any observational images of the large-scale structure of the Universe are flat (two-dimensional) views of three-dimensional structures. Three-dimensional maps of galaxy distributions show walls and filaments, and there are now several methods for quantifying filamentarity as well as tracking individual filaments. At the same time, technologies for producing additive materials lead to the formation of complex microstructures and deformations. Interfaces with different geometries and mechanical textures appear.

In both systems considered, a hierarchy of scales is observed. In this case, multiscale modeling is often used. Its idea is that system characteristics obtained at lower

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scales are used as inputs at a larger-scale level (Luscher et al., 2010). It should be noted that some early models of the formation of the large-scale structure of the Universe, such as models of hot dark matter, led to a top-down scenario in which excess energy on large scales resulted in a Universe filled with giant sheets extending over tens to hundreds of megaparsecs. Now there is a concept that dark matter is cold, and that the structure formed in an upward fashion: smaller structures form before larger ones (Bond et al., 2010).

Thus, in the case of using a local microstructure to restore the macrostructure and using the resulting solution of the microstructure as initial conditions for the macrostructure, there is a need to formulate and solve the corresponding boundary value problem.

2 Anisotropy of natural conglomerations

As already mentioned, it is the complex multi-level nature of some natural distributions of particles (mass) in a wide range of scales makes it difficult and sometimes almost impossible to directly observe their volumetric structure (as well as various types of structuring in it). Physical and mathematical modeling and, on its basis, parameterization of various structure formations, in this regard, it operates either with purely theoretical categories (statistical - distribution functions of groups of particles, Voronoi figures, graphs, etc.) or indirect characteristics, such as, for example, structural and form factors extracted from data on measuring scattering cross sections of external radiation. Both of them, along with the advantages that make it possible to obtain at different levels of accuracy the necessary information about the internal structure of conglomerations, also contain elements of inaccuracy, which in some cases can lead to very significant discrepancies with reality in the assessment of internal local parameters (Gerasymov, 2015; Gerasymov, 2024).

This problem stands in an especially contrasting way to the study of stellar conglomerations in the Universe. We have here a problem that requires, to be resolved, the establishment of connections between the characteristics of mass distributions observed in various planes and their volumetric prototypes. (Gerasymov & Kudashkina, 2022; Gerasymov & Kudashkina, 2024). Essentially the same problem exists in the structural analysis of many partial conglomerations that make up condensed matter in different phase states. In the latter case, it is resolved using the scattering of external radiation along the cross sections of which structural factors are reconstructed, the Fourier inversion of which with controlled accuracy provides information about an adequate measure of the state-binary distribution function (we do not dwell on the details of this method, which allows us to control the inaccuracies associated with non-uniform convergence of Fourier series when solving the inverse problem). The situation is more complicated in the case when it is necessary to restore and parameterize the distribution of masses and elements on the cosmic scale of the Universe, the scattering of which, carrying information about objects, is carried out in a natural way from a system of sources structured according to a multi-level principle. In this case, all that can be offered for a more or less adequate assessment of the distributions of matter is the topological relationship between three-dimensional objects and

their projections onto flat surfaces (which speculatively coincide with the planes of experimental observations).

It is one of these approaches, operating with multilevel analysis, that we propose as a toolkit in this work.

3 Theoretical description methods

In problems of structural analysis of global multi-level conglomerations on a cosmic scale, unfortunately, the subjects involved are not the concentration field $c(P,t)$, but rather the field of the surface of its level. Therefore, in cases where this is possible, it is natural and necessary to formulate problems with free boundaries in terms of these defining quantities, despite the significant complication of the differential equation, initial, and boundary conditions. Under these conditions, we present the Dirichlet problem (Krantz, 1999) for the level surfaces of the concentration field with source w.

If the Dirichlet boundary condition is specified on a known part $S(t)$ of the bounded surface of the volumetric domain $W(t)$, then to determine the concentration field $c(P,t)$ and the free surface we obtain a system of equations:

$$\begin{gathered}
\operatorname{div}(K \operatorname{grad} c) - c_t = f(c) - w, \quad P \in \Omega(t), \quad t > 0 \\
c(P,0) = c_0(P), \quad P \in \overline{\Omega}(0) \\
c(P,t) = \varphi(P,t), \quad P \in S(t), \quad t > 0 \\
c(P,t) = 0, \quad K\frac{\partial c}{\partial n}(P,t) = 0, \quad P \in \Gamma(t), t > 0
\end{gathered} \tag{1}$$

The level surface of the scalar concentration field $c = c(P,t)$ is defined in implicit form, between, a one-parameter family of level surfaces $\Phi(P,c,t)$, where $\Phi(P,c,t) = c(P,t) = c$. If the Dirichlet boundary condition is such that the region $W(t)$ is stratified on an iso-subsurface, then the concentration c will be considered a monotonic function of one of the coordinates, for example, z, for fixed values of x, y, t. This makes it possible, using a special transformation (Mises transformation) (Alderson, 2007) $c(x,y,z,t) \leftrightarrow z(x,y,c,t)$, to move from the problem of determining the field of concentration $c(x,y,z,t)$ to the problem of determining the field of its level surfaces $z(x,y,c,t)$. Such a transformation, by the way, allows us to get rid of the need to determine the free surface since the concentration in this case acts as an independent variable and the value $c = 0$ corresponds to this surface $\Gamma(t) : z = z(x,y,0,t)$. In this case, the following boundary value problem $z(x,y,c,t)$ can be formulated for level surfaces.

$$\begin{gathered}
Az = \frac{\partial z}{\partial t} + [f(c) - w(z)]\frac{\partial z}{\partial c} \\
\Omega_c(t) = \{(x,y) \in D; 0 < c < \overline{c}(x,y,t)\}, \quad t > 0 \\
z(x,y,c,0) = z_0(x,y,c), \quad x,y,c \in \overline{\Omega}(0) \\
z(x,y,c,t) = z_s(x,y,c,t) \\
x,y \in D, \quad c = \overline{c}(x,y,t), \quad t > 0
\end{gathered} \tag{2}$$

where $z = z_s(x, y, t)$ is the equation of a given and, thus, considered known surface $S(t)$; $z_n = \frac{\partial z}{\partial n}$, $z_c = \frac{\partial z}{\partial c}$; n – outer normal.

$$Az = \nabla_\tau (K \nabla_\tau z) - \frac{\partial}{\partial c}\left(K \frac{(\nabla z)^2}{\partial z / \partial c} \right),$$

∇_τ -– tangential part of the operator. When setting up problem (2), we used formulas for differentiating inverse functions, which lead to the following relations:

$$\operatorname{div}(K \operatorname{grad} c) = -\frac{Az}{z_c}, \frac{\partial c}{\partial t} = \frac{z_1}{z_c}, \frac{\partial c}{\partial t} = -\frac{z_1}{z_c}\frac{\partial c}{\partial n} = -\frac{z_n}{z_c}.$$

In the simplest one-dimensional case, the problem of determining the field of concentrations and the free surface $\Gamma(t)$ relative to the field of the level surface $z(c, t)$ is written in the form

$$\begin{gathered} K\frac{\partial^2 c}{\partial z^2} - \frac{\partial c}{\partial t} = f(c) - w, \quad 0 < z < z(t), \quad t > 0 \\ c(z, 0) = c_0(z), \quad 0 < z < z(0) \\ c(0, t) = \overline{c}(t), \quad t > 0, \quad (z(t), t) = 0 \end{gathered} \tag{3}$$

and

$$\begin{gathered} K\frac{\partial c}{\partial n}(z(t), t) = 0, \quad t > 0 \\ Az = \frac{\partial z}{\partial t} - [f(c) - w]\frac{\partial z}{\partial c}, \quad 0 < c < \overline{c}(t), \quad t > 0 \\ z(c, 0) = z_0(c), \quad 0 < c < \overline{c}(t), \quad z(\overline{c}(t), t) = 0, \quad t > 0 \end{gathered} \tag{4}$$

where $f(c) : f_c(c) > 0$ is the sink function; $\overline{c}(t)$ – monotonically non-decreasing function of time

$$Az = -\frac{\partial}{\partial c}\left[K \frac{1}{\partial z / \partial c} \right] = K\frac{z_c}{z_c^2} \tag{5}$$

The proposed approach is adequate to the task in the case where operator A is monotonic and unique. To establish the monotonic property of operator A, we use the well-known first Green formula

$$\int_0^{\overline{c}(t)} v Az \, dc = -K\frac{v}{z_c} + K\int_0^{\overline{c}(t)} \frac{v}{z_c} \, dc \tag{6}$$

Putting in (6) $v = z_2 - z_1$ and the first time $z = z_1$, and the second $z = z_2$, where $z_1 = z_1(c, t), z_2 = z_2(c, t)$ are arbitrary twice continuously differentiable functions defined on $(0, \overline{c}(t))$, we obtain two equalities

$$\int_0^{\overline{c}(t)} (z_2 - z_1) Az_1 \, dc = -K\frac{z_2 - z_1}{z_{1c}} + K\int_0^{\overline{c}(t)} \frac{(z_2 - z_1)_c}{z_{1c}} \, dc \tag{7}$$

$$\int_0^{\overline{c}(t)} (z_2 - z_1) A z_2 \, dc = -K \frac{z_2 - z_1}{z_{2c}} + K \int_0^{\overline{c}(t)} \frac{(z_2 - z_1)_c}{z_{2c}} \, dc \tag{8}$$

Subtracting (7) from (8), we obtain the following relation:

$$\int_0^{\overline{c}(t)} (Az_2 - Az_1)(z_2 - z_1) \, dc =$$

$$= -K \left[(z_2 - z_1) \left(\frac{1}{z_{2c}} - \frac{1}{z_{1c}} \right) \right] + K \int_0^{\overline{c}(t)} (z_2 - z_1)_c \left(\frac{1}{z_{2c}} - \frac{1}{z_{1c}} \right) dc$$

which, after a simple transformation of the integrand in the second term on the right side, is written as

$$\begin{aligned} &\int_0^{\overline{c}(t)} (Az_2 - Az_1)(z_2 - z_1) \, dc = \\ &= -K \left[(z_2 - z_1) \left(\frac{1}{z_{2c}} - \frac{1}{z_{1c}} \right) \right] - \frac{K}{2} \int_0^{\overline{c}(t)} \frac{(z_2 - z_1)_c^2}{z_{1c} z_{2c}} \, dc \end{aligned} \tag{9}$$

Let now $z_1 = z_1(c,t)$, $z_2 = z_2(c,t)$ be two different solutions to problem (4). Then, due to the boundary conditions, the first term in (9) is equal to zero, and the integrand in the second term, due to the monotonicity of the functions $z_1(c,t)$ and $z_2(c,t)$ in $c(z_{1c}, z_{2c} > 0$, is positive. Consequently, the first Green formula for the operator of problem (4) will finally be written in the form

$$\int_0^{\overline{c}(t)} (Az_2 - Az_1)(z_2 - z_1) \, dc = -\frac{K}{2} \int_0^{\overline{c}(t)} \frac{(z_2 - z_1)_c^2}{z_{1c} z_{2c}} \, dc \tag{10}$$

Since the right-hand side of equality (10) is not positive, this implies the monotonicity of operator A on the set of solutions to problem (4). Using the monotonicity property of operator A, we prove the uniqueness of the solution to problem (4). To do this, using the differential equation of the problem, we eliminate Az_1 and Az_2 from (10). We have

$$\begin{aligned} &\int_0^{\overline{c}(t)} (Az_2 - Az_1)(z_2 - z_1) \, dc \\ &= \int_0^{\overline{c}(t)} (z_2 - z_1) \{ (z_{2t} - z_{1t}) - f(c)(z_{2c} - z_{1c}) + w(z_{2c} - z_{1c}) \} \\ &= \frac{1}{2} \int_0^{\overline{c}(t)} \{ (z_2 - z_1)_t^2 + w(z_2 - z_1)_c^2 \} \, dc - \int_0^{\overline{c}(t)} f(c)(z_2 - z_1)_c^2 \, dc \end{aligned} \tag{11}$$

Integrating the last integral by parts, taking into account the boundary conditions of problem (4) and the properties of the function *f(c)*, we rewrite (11) in the form

$$\int_0^{\overline{c}(t)} (Az_2 - Az_1)(z_2 - z_1)\, dc = \quad = \frac{1}{2} \int_0^{\overline{c}(t)} \{(z_2 - z_1)_t^2 + w(z_2 - z_1)_c^2\}\, dc + \\ + \int_0^{\overline{c}(t)} f_c(c)(z_2 - z_1)^2\, dc \tag{12}$$

As it is easy to see, the right-hand side of the resulting equality is non-negative. However, it was said above that the right-hand side is not positive (see (10)). From the resulting contradiction it follows that $z_1(c,t) = z_2(c,t)$.

Thus, we come to the following main conclusion: if the source function and the sink function *f(c)* increases monotonically, then the solution to the one-dimensional Dirichlet problem (4) for level surfaces is positive and unique. And the approach proposed above as a whole can be used to solve the problem of multi-level structural analysis.

4 Similar features and differences

In the context of the morphological similarity of the large-scale structure of the Universe and the cluster structure of granular materials, several aspects can be noted: 1) organization by concentration levels – in both systems (the Universe and granular materials) an organization by levels of concentration of matter is observed. In space, this relates to the distribution of matter and galaxies across different structural levels, while in granular materials it may refer to the formation of clusters or cellular structures of particles; 2) monotonic increase in the source function and sink function – the conclusion about the unique and positive solution to the Dirichlet problem is associated with a monotonic increase in the functions, which indicates that the formation of the structure occurs by certain patterns. In the case of cosmic structure, this may be due to gravitational dynamics and the evolution of galaxies, and in the case of granular materials, to mechanical or chemical processes; 3) uniqueness and certainty of the structure – in both cases, both in the Universe and in granular materials, there is a certain uniqueness of the structure, which is formed based on interactions between the elements of the system. This may be reflected in similar forms of organization such as clusters, mosaic structures, or cellular formations; 4) analogies in formation processes – the conclusion about the similarity of the behavior of these systems and the processes that create such a structure can be justified by analogies in the formation mechanisms. For example, gravitational processes in space and particle aggregation in granular materials may have similar principles, although the scales and conditions are different.

It is necessary, however, to take into account significant differences in the physical processes and characteristics of the systems under consideration. The main differences include: 1) scale and size and, as a consequence, various formative processes; 2)

physical processes – for the Universe these are gravitational interactions of galaxies, while in granular materials the processes of aggregation, coagulation, adhesion, and dispersion are important; 3) composition and properties of matter – dark matter and energy also play a major role in the Universe. In granular materials, not only physical but also chemical properties are important, as well as component composition, particle sizes, etc.

5 Conclusions

The Dirichlet problem is used here to describe the boundary between regions with different concentrations of matter or different structures. Solving this problem makes it possible to identify the features of the internal structure and organization of the conglomerate by determining the shape of the surface level of the concentration field with the source.

The conclusion about the positive and unique solution to the one-dimensional Dirichlet problem with a monotonic increase in the source function and sink function *f(c)* can be interpreted as follows: 1) a positive solution means that there is a single level surface of the concentration field that satisfies the boundary conditions and characteristics of the source and sink functions. In the context of structural analysis of conglomerates, this may mean that there is a certain shape or structure corresponding to the distribution of masses and elements in the area of interest; 2) the uniqueness of the solution indicates that there are no other alternative forms or structures that could satisfy the given conditions. In the context of the analysis of cosmic or micromechanical conglomerates, this means that only a certain configuration or organization of matter corresponds to given source and sink functions.

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