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Towards the classification of mass distributions with complex morphology in terms of topological parameters (from granular matter to cosmology)

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In this paper, we briefly present, from a practical point of view, different microstructure stereological parameters (Euler–Poincaré characteristics) allowing a quantitative description of geometrical features of a material with a complex internal morphology within meso- and macro-scales.

Considering conditionally the visible distribution of super-galaxies (voids and filaments) as a "microstructure", the proposed method can be applied to the parametrization of the distribution of matter in the Universe.

Keywords: Universe, granular materials, filaments, voids, stereological parameters, Euler–Poincaré characteristics

1 Introduction

It is known that our Universe is homogeneous only on very large scales ($L \gtrsim 200$ Mpc). As the scale decreases, a pronounced hierarchical structure is visible. In particular, super clusters of galaxies ($L \backsim 100$ Mpc) form relatively thin walls ($\backsim$10–20 Mpc), limiting empty regions (more precisely, regions of lower concentration of luminous matter), forming a large-scale structure of the Universe (Pustilnik et al., 2019).

As already noted in (Gerasymov & Kudashkina, 2022), there are certain analogies between the Universe, its large-scale structure, and the properties of granular systems. This allows us to consider some of the methods applied to the study of granular materials for the analysis of the cellular structure of the Universe. In particular, to characterize the three-dimensional geometry of the distribution of objects (super clusters of galaxies, voids) based on incomplete observations of one-dimensional and two-dimensional "slices" in this space. For example, such longitude slices are presented by Jones et al. (see Figs 8 and 9 in that paper).

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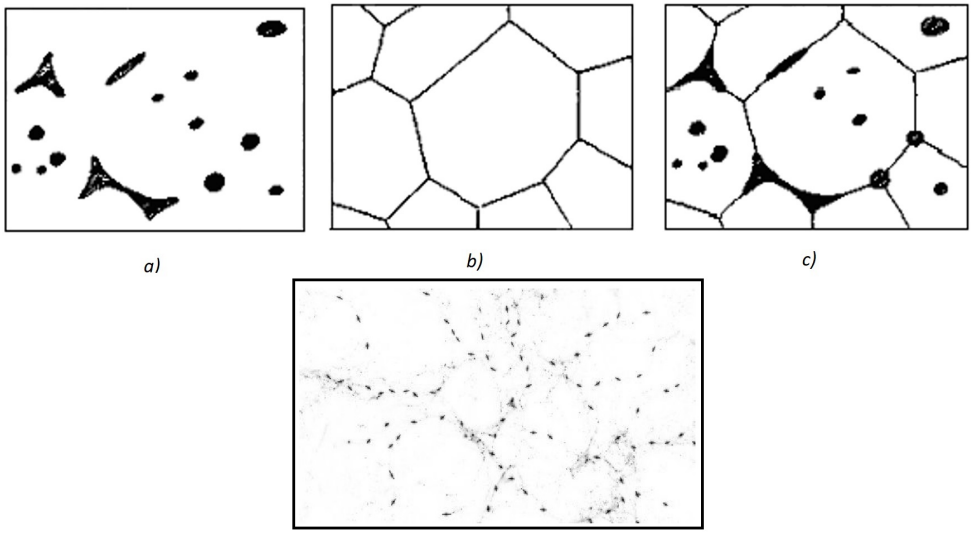


Figure 1 Examples of sections through three-dimensional structures: a) structure with two phases (S) white and (P) black (one interface (S/P), dimension 2); b) structure with one phase (S) (one interface (S/S), dimension 2, and one linear interface ($S/S/S$), dimension 1); c) structure with two phases (S) white and (P) black (two interfaces (S/S) and (S/P), dimension 2 and two interfaces ($S/S/S$) and ($S/S/P$), dimension 1). The bottom figure is shown as an illustration of the cellular structure of the universe. Image courtesy of ESO (https://www.eso.org/public/news/eso1438/).

In the present work, an attempt is made to extend the mathematical model constructed with the help of stereological characteristics for micromechanical systems to the cellular structure of the Universe.

The task is rather complicated for at least two reasons. First, the 3-d microstructure is generally not observable directly (we are leaving aside here indirect methods, such as analysis of the structure factors of scattering of external radiations). Among the direct methods, plane cross-sections or, at most, serial plane sections can be observed quite often. The aim of stereology is to extract 3-d information starting from 2-d measurements. Second, a statistical problem arises: is the available specimen representative of the whole structure?

In this paper, only the first mentioned problem will be considered. For example, a plane section through a porous structure can give a picture like (a) or (c) in Fig. 1. Intuitively, from this picture one can define, for example, a relative amount or a typical size of the porous phase. We shall see how stereology relates three-dimensional parameters to measurements on plane sections. Another fundamental problem is to estimate the degree of interconnection of the porous phase. One plane section is not sufficient in this case, but several sections in the material separated by short distances (serial sectioning) can give an answer to that question.

This paper will focus mainly upon a practical level of the stereology (relations between microstructure and properties of sintered or granular materials on micro-scales and geometry of distribution of super clusters of galaxies on mega-scales).

2 Classification (basic parameters)

The construction of theoretical approaches to the description and parameterization of the distributions of discrete masses in granular matter and foam is an urgent task of modern physics, especially in terms of establishing a connection between the types of structural organization and the manifested macroscopic properties (Gerasymov, 2015). It is interesting to note that essentially the same task arises in cosmology, where the observed distributions of matter have a largely similar morphology (Conselice et al., 2016). Apparently, it would be an exaggeration to consider that the distribution of masses in the sector (more precisely, in the plane) of observation directly corresponds to the law of self-similarity. Rather, we can expect that the integral distribution (total) will have such a property. By specifying the latter in the form of a homogeneous (power-law) amplitude-modulated function $\phi(M, z)$, we obtain the possibility of a two-parameter classification of the observed distributions.

$$f(z) = \int_{M_1}^{M_2} \phi(M, z)\, dM. \tag{1}$$

The entanglement of the structure in the listed systems requires the development of various alternative methods for its description and adequate parameterization. On this path, a fairly large number of approaches have been developed, which, given the complexity of the objects described, are not universal. It seems that the descriptions of topological methods of systems with complex internal morphology can be a good candidate because, apart from some detailing of complex distributions, they provide sufficient information for their classification and, in the limit of elementary distributions, bring us to known exact results.

To introduce the different quantities measured by stereology we assign X the object to be measured (phase, interface, intruders, etc) and w the measurement (w can be a number of points, a length, a plane surface, a curved surface, a volume or a number of connectivity and it will be noted further as P, L, A, S, V or N, respectively.

We will write $w(X)$ for global analysis (Serra, 1982; Aherne & Dunnill, 1982) (if the set X is totally included in the frame of measurement). In the case of local analysis (if the set X is partially included in the frame of measurement as it is the case in materials science because the analysis of a sample is performed on a small part of the structure) an index is added to the notation $w(X)$. This index characterizes the space of measurement of the parameter $w(X)$, namely: X is P for punctual, L for linear, A for surface and V for volume.

3 Euler–Poincaré characteristics

The global parameters that can be defined by an image without any restrictive hypothesis on the structure can be simply expressed using an unique stereological quantity called the connectivity number or Euler–Poincaré characteristic (Kelley, 2017; Ghrist, 2014). Let X be a closed set (containing its boundary). The connectivity number of X in R is defined by induction with respect to the space dimension n.

Table 1 Basic parameters.

Space	Parameters							
R^3	$V(X) = N_0(X)$	δV	$S(X) = 4N_1(X)$	δA	$M(X) = 2\pi N_2(X)$	δL	$N_3(X)$	
R^2	$A(X) = N_0(X)$	δA	$L(X) = \pi N_1(X)$	δL	$N_2(X)$			
R^1	$L(X) = N_0(X)$	δL	$N_1(X)$					
R^0	$N_0(X)$							

Table 1 shows the following basic parameters:

• the connectivity number $N_0(X)$, in R^0, is the number of points of X belonging to the intersection of X with a superimposed grid of points;

• the connectivity number $N_1(X)$, in R^1, is the number of segments cut in X by a straight line;

• the connectivity number $N_2(X)$, in R^2, is the number of connected particles in X minus the number of holes contained in X;

• the connectivity number $N_3(X)$, in R^3, is related to the number and the genus of its different connected components according to the formula:

$$N_3(X) = s - \sum_s \wp, \tag{2}$$

where s is a number of distinct surfaces, and $\wp$ is a genus of each surface. The genus of a surface is the maximum number of closed curves that can be drawn on this surface reserving its connectivity. It is important to note that a serial analysis of the structure is necessary to get $N_3(X)$.

Furthermore Table 1 gives the relations between the selected connectivity numbers and the geometrical parameters.

4 Local analysis of structures: metric and topological properties

Usually, whole three-dimensional structures are not directly accessible to measurements: a little pan of the structure is analyzed inside a frame. If the hypothesis of stationary state is made when a structure is analyzed in local conditions, then the parameters are divided by the unit of the space where they are measured in order to make them independent of the size of the frame.

The metric parameters that can be estimated from plane sections without any hypothesis based on the shape of the phases are as follows:

• the volumetric fraction $V_V(X)$ of the phase, which is the total volume of phase X contained in one unit volume of micro-structure;

• the specific surface area per unit volume $S_V(X/Y)$ which is the total area of the interface per unit volume of micro-structure (denoted as (X/Y);

• the integral of mean curvature per unit volume $M_V(X/Y)$. At each point of the interface (X/Y) the principal normal curvatures allow us to define a mean curvature H for an element of surface dS. The integral of mean curvature $M_Y(X/Y)$ is obtained by integrating HdS over the whole surface (X/Y) contained in one unit volume and dividing by the unit volume of micro-structure.

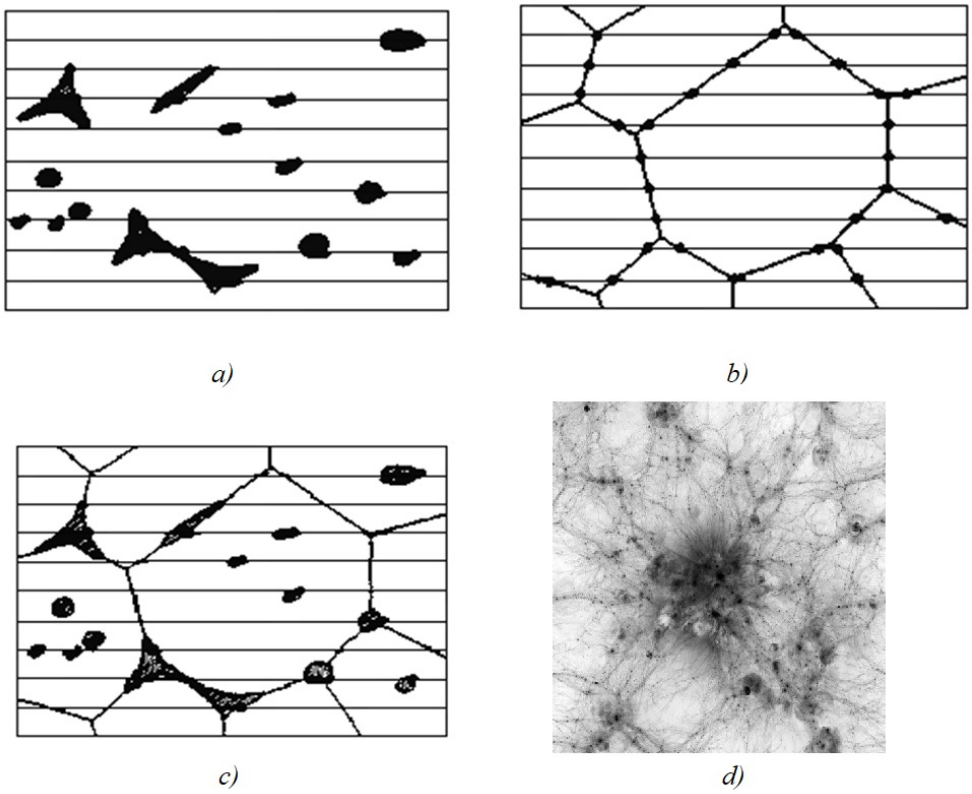


Figure 2 Determination of topological and metric parameters of the given structures. The right image (d) is a very detailed simulation of large scale structure. The region shown is about 300 million light-years across. Image courtesy of ESO (https://www.eso.org/public/news/eso1438/).

If the geometry of the structure is known *a priori*, this allows us to make some hypothesis giving access to other parameters. For example, if there are some linear features in the structure, a stereological relation gives the length of these features per unit volume from measurements in a section.

To determine the volumetric fraction V_V we can write down:

$$V_V = A_A = L_L = \frac{N_0}{P}, \tag{3}$$

where N_0 is an Euler–Poincaré characteristic in R^0 space (number of test-points hitting the phase) and P is a total number of test points. To estimate the perimeter L, let S be a set of oriented parallel lines in direction Δ regularly separated by distance ∂L, and $N_1(X)$ be the total number of points (black dots in Fig. 2) of the boundary of X intercepted by these lines. Thus the perimeter $L(X)$ of X can be estimated by the relation: $L(X) = \pi N_1(X)\partial L$. The product $N_1(X)\partial L$ is called diametric variation of X in direction Δ.

In order to determine of the specific surface area per unit volume S_V we can use the formula

$$S_V = \frac{4}{\pi} L_A = 4 \frac{N_1}{L}, \tag{4}$$

where N_1 is an Euler–Poincaré characteristic in R^1 space (number of exits from the phase along the line of analysis), and L is the total length of the line of analysis. For instance, for the data presented in Fig. 2 we have: (a) $S_V(S/P) = 4$; (b) $S_V(S/S) = 4$; (c) $S_V(S/P) = 4$.

We can also determine the integral of mean curvature M_V given by the relation

$$M_V = 2\pi \frac{N_2}{A} = 2\pi \frac{N_2^+ - N_2^-}{A}, \tag{5}$$

where N_2 is the Euler–Poincaré characteristic in R_2 space; N_2^+ is the convexity number and N_2^- is the concavity number, which can be determined by sweeping a straight line across the area of analysis. The positive counts correspond to exits from the phase with a convex arc and the negative ones with a concave arc.

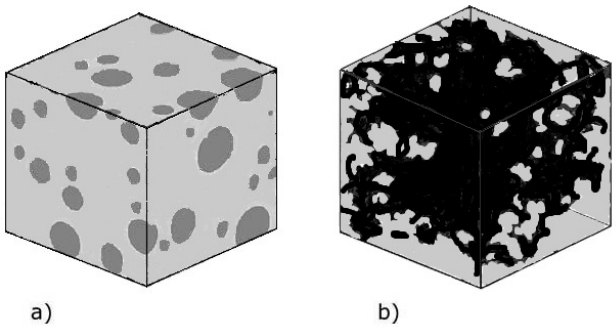


Figure 3 Euler–Poincaré characteristic and interconnection of a phase: a) totally dispersed structure $N_3 = s$ (in this case, the connectivity number is the physical number of particles; b) totally interconnected structure $N_3 = 1 - g$ (in this case, the connectivity number is directly linked to the genus).

The Euler–Poincaré characteristics are topological parameters (invariant under homotopic condition) and they are accessible only in their own space. As mentioned above, there is a relation between the genus and the connectivity number N_3. Unfortunately, this relation is not simple and the practical meaning of N_3 can change with the structure. Anyway, it is an important parameter because it is the only one that gives an idea of the degree of interconnection of a structure (permeability of porous media). We recall that the genus of a surface is the maximum number of closed curves that can be drawn on this surface preserving its convexity, a relation between the genus and Euler–Poincaré characteristic could be described by (2). Figure 3 shows schematically an Euler–Poincaré interconnection algorithm in the 3d case.

5 Conclusions

In this paper, some "recipes" were given to describe directly and quantitatively the structure of materials using practically only one fundamental parameter: the Euler–Poincaré characteristic. The quantitative image analysis can also be carried out using

a mathematical morphology approach (Mouton, 2009; Baddeley & Jensen, 2004; Liu, 2000). The distribution of matter in the Universe on the considered scales is extremely inhomogeneous and anisotropic. An insufficiently defined density field makes it difficult to use standard methods using a Cartesian-like coordinate description. The information extracted from observations using cross sections is limited and requires large-scale refinements (the latter show the complex volumetric nature of the density field), limited efforts to process a significant number of observations, and resulting cross sections. But in essence, the problem turns out to be largely isomorphic to the structural analysis of porous, friable media, foamy structures and (or) their mixtures with a complex topology of the distribution of elements and internal structuring. The closeness in the formal description of objects of a different nature leads to the same solutions to the problem of restoring their volumetric organization.

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