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# Angular momentum to a distant observer

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In the first observation of gravitational waves GW150914 by LIGO and Virgo [1], a binary black hole of initial black hole masses 36 units and 29 units merge into a final black hole with 62 units of mass, during which exactly 3 units of mass radiated in gravitational waves. Measuring the angular momentum carried away in gravitational waves turned out to be more challenging. This has been an open problem in classical general relativity due to "supertranslation ambiguity" since the 1960s. Two distant observers of the same system may record different amount of "radiated angular momentum". There have been a lot of efforts to resolve the supertranslation ambiguity. One approach is to find a new definition of angular momentum that would yield the same amount of radiated angular momentum for any observer. Last year, the first such ambiguity free definition of angular momentum was discovered and it is the purpose of this note to discuss this new definition.

## 1 Angular momentum in classical mechanics and special relativity

In classical mechanics, the angular momentum of a particle is defined as

$$J = m\mathbf{r} \times \mathbf{r}'.$$

If we translate the origin (or the coordinate system) from $O$ to $\tilde{O}$, the angular momentum $J$ gets shifted by the linear momentum $\mathbf{p}$

$$J_{\tilde{O}} = m(\mathbf{r} - \mathbf{a}) \times \mathbf{r}' = J_O - \mathbf{a} \times \mathbf{p}.$$

Special Relativity can be formulated in terms of the geometry of the Minkowski spacetime $\mathbb{R}^{3,1}$. The continuous symmetry of $\mathbb{R}^{3,1}$ consists of

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x^i}, \quad x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, i < j, \quad t\frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t},$$

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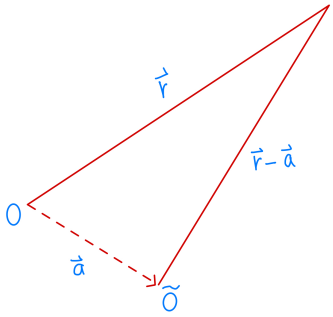



**Figure 1** Angular momentum gets shifted by the linear momentum.

which generates the time translation, spatial translations, rotations, and Lorentz transformations, respectively.

Noether's Theorem states that each continuous symmetry corresponds to a conserved quantity. For a particle $\gamma$ in $\mathbb{R}^{3,1}$, we have

$$\begin{aligned}
&\text{energy} && e = \langle \gamma', \frac{\partial}{\partial t} \rangle \\
&\text{linear momentum} && p_i = \langle \gamma', \frac{\partial}{\partial x^i} \rangle \\
&\text{angular momentum} && J_{ij} = \langle \gamma', x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \rangle, i < j \\
&\text{center of mass} && C_i = \langle \gamma', t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t} \rangle.
\end{aligned}$$

If we translate the coordinate system $t \mapsto t + \alpha^0, x^i \mapsto x^i + \alpha^i$, the energy and linear momentum are the same but the angular momentum and center of mass get shifted

$$J'_{ij} = J_{ij} + \alpha_i p_j - \alpha_j p_i, i < j \tag{1.1}$$

$$C'_i = C_i + \alpha^0 p_i + \alpha^i e. \tag{1.2}$$

## 2 Distant observer in the Bondi–Sachs formalism in general relativity

In general relativity, an idealized distant observer is situated at future null infinity $\mathscr{I}^+$, where light rays approach along null geodesics. Descriptions of $\mathscr{I}^+$ of an isolated gravitating system include: Bondi–Sachs coordinates (Bondi et al. 1962 [6], Sachs 1962 [36]), Penrose conformal compactification (Penrose 1965 [33]) and Christodoulou–Klainerman 1993 [13], etc.

Bondi–Sachs and Penrose formalisms are essentially equivalent; both of them imply the spacetimes have "peeling" – an idealized property that may not hold in general, as demonstrated in the work of Christodoulou–Klainerman (see also L. Bieri [4]). In this note, we describe null infinity and distant observers using the Bondi–Sachs formalism. Our result can be extended to "polyhomogeneity" (Chrusciel–MacCallum–Singleton, 1995 [15]) $\mathscr{I}^+$ and the Christodoulou–Klainerman setting, see [11].

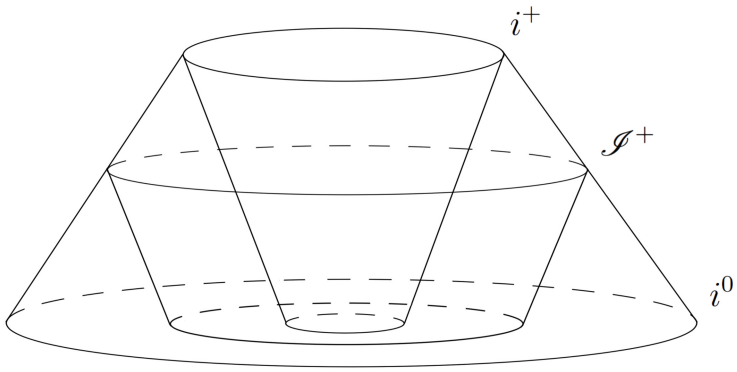



**Figure 2** The future null infinity $\mathscr{I}^+$ extends from spacelike infinity $i^0$ ($u = -\infty$) to timelike infinity $i^+$ ($u = +\infty$).

Recall the Schwarzschild spacetime in Eddington–Finkelstein coordinates:

$$-(1 - \frac{2m}{r})du^2 - 2dudr + r^2\sigma_{AB}dx^A dx^B, \tag{2.1}$$

where $\sigma_{AB}dx^A dx^B = d\theta^2 + \sin^2\theta d\phi^2$, $u$ is the "retarded time" and $r$ is the luminosity distance from the black hole. The future null infinity $\mathscr{I}^+$ corresponds to $r = \infty$. $u = +\infty$ corresponds to future timelike infinity $i^+$ and $u = -\infty$ corresponds to spacelike infinity $i^0$.

Bondi and his collaborators postulated a coordinate system (Bondi–Sachs) in which the metric tensor of the spacetime is given by

$$-UVdu^2 - 2Ududr + r^2 h_{AB}(dx^A + W^A du)(dx^B + W^B du). \tag{2.2}$$

The spacetime is assumed asymptotically flat in the sense that $U, V \to 1$, $h_{AB} \to \sigma_{AB}$, $W^A \to 0$ as $r \to \infty$ and $\det h_{AB} = \det \sigma_{AB}$ (determinant condition) for $r$ large. Moreover, outgoing radiation condition is imposed so that all metric coefficients $U, V, h_{AB}, W^A$ can be expanded into power series of $\frac{1}{r}$. The assumption is relaxed to "polyhomogeneity" by Chrusciel–MacCallum–Singleton [15].

Bondi et al. [6, 36] found that the physics of gravitational field are encoded in the coefficients of power series expansion. Energy and linear momentum are well-understood in the Bondi–Sachs formalism. In particular, the expansion of $V$ in (2.2)

$$V = 1 - \frac{2m(u,x)}{r} + O(r^{-2})$$

gives the *mass aspect* $m(u,x)$ on $\mathscr{I}^+$.

The Bondi–Sachs energy momentum $(E, P^k), k = 1, 2, 3$ associated with a constant $u$ slice is

$$E(u) = \int_{S^2} 2m(u,\cdot), \quad P^k(u) = \int_{S^2} 2m(u,\cdot)\tilde{X}^k,$$

where $\tilde{X}^1 = \sin\theta\cos\phi, \tilde{X}^2 = \sin\theta\sin\phi, \tilde{X}^3 = \cos\theta$ are the restriction of standard coordinate functions of $\mathbb{R}^3$ to $S^2$. The positivity of $E(u)$ was proved by Schoen–Yau 1982 [37] and independently by Horowitz–Perry 1982 [24].

In addition, $E(u)$ satisfies the Bondi energy loss formula [5]:

$$\frac{d}{du}E(u) = -\frac{1}{4}\int_{S^2} |N_{AB}|^2 \leq 0, \tag{2.3}$$

where $N_{AB} = \partial_u C_{AB}$ is called the *news* and $C_{AB}$ is the shear that appears in the expansion of $h_{AB}$ in (2.2):

$$h_{AB} = \sigma_{AB} + \frac{C_{AB}}{r} + O(r^2).$$

Both $C_{AB}$ and $N_{AB}$ are traceless with respect to $\sigma_{AB}$ because of the determinant condition $\det h_{AB} = \det \sigma_{AB}$. Equation (2.3) is indeed one of the first theoretical studies of gravitational radiation. In the linear case, the formula is due to Trautman [38, 39] and the nonlinear case due to Bondi [5].

The definition of angular momentum turns out to be more subtle largely because of the convergence of the expansions above and the choice of the Bondi–Sachs coordinate. The corresponding Killing fields are the rotation fields, which are of higher order near infinity in comparison to the translating Killing fields used in the definition of the energy and linear momentum.

As a result, a very precise comparison with the Minkowski spacetime is needed in order to find an appropriate definition of angular momentum. In the literature, there are many approaches to define angular momentum (Hamiltonian, spinor-twistor, Komar type etc.), leading to different definitions such as Newmann–Penrose [30], Winicour–Tamburino [43], Bramson 1975 [7], Ashtekar–Hansen 1978 [2], Penrose 1982 [31], Ludvigsen–Vickers 1983 [27], Dray–Streubel 1984 [18], Moreschi 1986 [29], Dougan–Mason 1991 [17], Rizzi 1997 [35], Chruściel-Jezierski–Kijowski 2002 [14], Helfer 2007 [21, 22], Barnich–Troessaert 2011[3], Hawking–Perry–Strominger 2017 [20] Javadinezhad–Kol–Porrati 2019 [23], Klainerman–Szeftel 2019 [26], Compère–Oliveri–Seraj 2020 [16], etc.

All these definitions are of the form

$$\int_{S^2} Y^A(N_A + \cdots),$$

where $Y^A$ is a rotation Killing field on $S^2$ and $N_A = \sigma_{AB}N^B$, called the *angular momentum aspect*[1], appears in the higher order expansion of $W^A$ in (2.2)

$$W^A = \frac{1}{2r^2}\nabla^D C^A_D + \frac{1}{r^3}\left(\frac{2}{3}N^A - \frac{1}{16}\nabla^A(C_{DE}C^{DE}) - \frac{1}{2}C_B{}^A\nabla^D C_D{}^B\right) + O(r^{-4}).$$

Each definition recovers the Kerr angular momentum in a certain coordinate system.

[1]It corresponds to $\Psi_1$ in Newman–Penrose formalism.

## 3 Supertranslation ambiguity

It turns out there is no preferred coordinate system (in particular no preferred $u$ coordinate) when radiation is present. For any smooth function $f(x)$ on $S^2$, the change of coordinates $u = \bar{u} + f(x)$ is called *supertranslation*. Geometrically, supertranslation changes the foliation of $\mathscr{I}^+$.

We decompose $f$ into

$$f = \alpha_0 + \sum_{i=1}^{3} \alpha_i \tilde{X}^i + S$$

where $\alpha_0 + \sum_{i=1}^{3} \alpha_i \tilde{X}^i$ corresponds to (4-dimensional) ordinary translation and $S$ is referred to as *pure supertranslation*.

For a supertranslation $u = \bar{u} + f(x)$,

$$\begin{aligned} \bar{m}(\bar{u}, x) &= m(\bar{u} + f, x) + \frac{1}{2}(\nabla^B N_{BD})(\bar{u} + f, x)\nabla^D f \\ &\quad + \frac{1}{4}(\partial_u N_{BD})(\bar{u} + f, x)\nabla^B f \nabla^D f + \frac{1}{4} N_{BD}(\bar{u} + f, x)\nabla^B \nabla^D f \\ \bar{C}_{AB}(\bar{u}, x) &= C_{AB}(\bar{u} + f(x), x) - 2\nabla_A \nabla_B f + \Delta f \sigma_{AB} \\ \bar{N}_{AB}(\bar{u}, x) &= N_{AB}(\bar{u} + f(x), x). \end{aligned} \tag{3.1}$$

Consequently, the total flux of energy

$$\delta E = E(+\infty) - E(-\infty) = -\frac{1}{4} \int_{-\infty}^{+\infty} \int_{S^2} |N_{AB}|^2$$

is supertranslation invariant and the energy radiated away is without ambiguity.

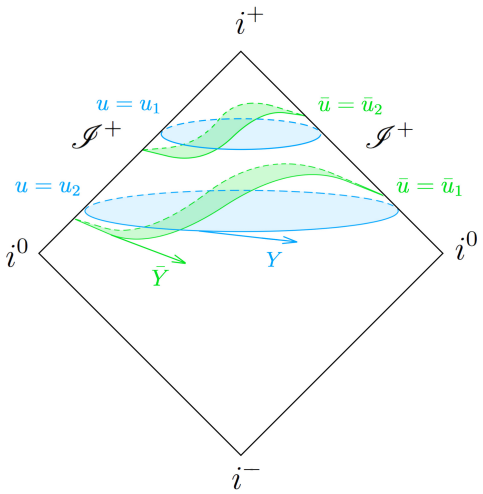



**Figure 3** A supertranslation that maps $\bar{u}$ cuts to $u = \bar{u} + f(x)$ cuts. $Y$ and $\bar{Y}$ are rotation BMS fields such that $Y$ is tangent to $u$ cuts and $\bar{Y}$ is tangent to $\bar{u}$ cuts.

On the other hand, under a supertranslation, the total flux of *classical angular momentum* of Dray–Streubel [18]

$$\tilde{J} = \int_{S^2} Y^A (N_A - \frac{1}{4} C_A{}^D \nabla^B C_{DB})$$

transforms according to

$$(\delta \tilde{J})_f - \delta \tilde{J} = -2 \int_{S^2} f \; Y^A \nabla_A (m(+) - m(-)),$$

where

$$m(\pm) = \lim_{u \to \pm\infty} m(u, x) \in C^\infty(S^2)$$

are the limits of the mass aspect at $i^+$ and $i^0$.

In particular, if $Y^A \nabla_A (m(+) - m(-))$ has any $\ell \geq 2$ spherical harmonics component, the total flux of the classical angular momentum $\delta \tilde{J}$ can assume any value by supertranslations.[2] According to Penrose [32], the very concept of angular momentum gets "shifted" by supertranslations and "it is hard to see in these circumstances how one can rigorously discuss such questions as the angular momentum carried away by gravitational radiation".

There were also efforts to eliminate supertranslation ambiguity by choosing special null foliations, in particular "nice sections"[3] by Moreschi 1986 [29] and "preferred cuts"[4] by Rizzi 1997 [35]. However, both work only in special cases[5].

## 4 Chen–Wang–Yau angular momentum and angular momentum at null infinity

The Chen–Wang–Yau quasilocal angular momentum [8, 9] was defined based on the theory of Wang–Yau quasilocal mass [41, 42]. The quasilocal formulation is essential in this work on angular momentum at null infinity. This was pointed out by Penrose in 1982 as he considered the definition of quasilocal mass and quasilocal angular momentum as his number one and number two unsolved problems in general relativity [32].

Given a spacelike 2-surface $\Sigma$ in a spacetime, we first extract the physical data $(\sigma, |H|, \alpha_H)$ where $\sigma$ is the induced Riemannian metric and $H$ is the mean curvature vector field on the surface. For each time function $X^0 = \tau$, we solve the following equations

$$\sum_{i=1}^{3} \partial_a X^i \partial_b X^i = \sigma_{ab} + \partial_a X^0 \partial_b X^0. \tag{4.1}$$

$X = (X^0, X^1, X^2, X^3)$ gives a configuration of $\Sigma$ in $\mathbb{R}^{3,1}$. We extract the reference data $(\sigma, |H_0|, \alpha_{H_0})$ from the image of $X$ in $\mathbb{R}^{3,1}$ accordingly. From these two sets of

[2] In fact, none of the definitions listed above were known to be supertranslation invariant.

[3] With zero Bondi–Sachs momentum.

[4] With zero electric part of the shear.

[5] $|\dot{\sigma}|^2 < \sqrt{\frac{27}{4}}$ and $|\Xi|^2 < 16$, respectively.

data, we introduce $\rho$ and $j$

$$\rho = \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}},$$
$$j = \rho\nabla\tau - \nabla[\sinh^{-1}(\frac{\rho div(\nabla\tau)}{|H_0||H|})] - \alpha_{H_0} + \alpha_H.$$

Integration of $\rho$ gives rise to the energy of this embedding $E(X)$ which depends on the two sets of data. The Euler–Lagrange equation by minimizing $E(X)$ among all possible isometric embeddings $X$ gives rise to the equation:

$$\nabla^a j_a = 0. \tag{4.2}$$

For surfaces near null infinity, the system of equations (4.1) and (4.2) has a unique solution, which can be calculated numerically. Transplanting a rotation Killing field $Y$ of $\mathbb{R}^{3,1}$ through this unique solution, we define the quasilocal angular momentum on $\Sigma$ as

$$\int_\Sigma j(Y^\top),$$

where $Y^\top$ is the component of $Y$ that is tangent to the image of $X$ in $\mathbb{R}^{3,1}$. The limit of quasilocal angular momentum defines the new total angular momentum at null infinity. This can be carried out for very general spacetimes. In particular, the peeling structure of null infinity is not needed in such calculation.

Effectively, in a Bondi–Sachs coordinate system, on a fixed $u$-slice, consider 2-surfaces $r = r_0$ and let $r_0 \to \infty$. The limit of the CWY quasilocal angular momentum at $\mathscr{I}^+$ was evaluated by Keller–Y.-K. Wang–Yau in [25].

Recall that the *classical angular momentum* (Dray–Streubel 1984 [18]) is

$$\tilde{J} = \int_{S^2} Y^A (N_A - \frac{1}{4} C_A{}^D \nabla^B C_{DB}).$$

In comparison, the limit of *CWY angular momentum* is

$$J = \int_{S^2} Y^A (N_A - \frac{1}{4} C_A{}^D \nabla^B C_{DB} - c\nabla_A m),$$

where $c$ is given by the decomposition of the shear tensor $C_{AB}$ and has never occurred in any previous definition of angular momentum. The decomposition of $C_{AB}$ is given by

$$C_{AB} = \nabla_A \nabla_B c - \frac{1}{2}\sigma_{AB}\Delta c + \frac{1}{2}(\epsilon_A{}^E \nabla_E \nabla_B \underline{c} + \epsilon_B{}^E \nabla_E \nabla_A \underline{c})$$

for $c$ and $\underline{c}$ of harmonic mode $\ell \geq 2$.

The term $c$ arises in the CWY angular momentum $J$ through solving the the system of equations (4.1) and (4.2) to obtain the gravitational ground state. In particular, the additional term in the definition of $J$ indeed comes from the reference term in the Hamiltonian theory.

**Theorem 1 (Chen–Keller–Wang–Wang–Yau, 2021)** *[10, 12]*

*Suppose $N_{AB}(u,\cdot) = O(|u|^{-1-\epsilon})$ as $|u| \to \infty$. Under a supertranslation $f = \alpha_0 + \alpha_i \tilde{X}^i + S$, the total flux of the CWY angular momentum*

$$\delta J(Y) = J(+\infty, Y) - J(-\infty, Y)$$

*transforms according to*

$$(\delta J)_f - \delta J = \alpha_i \epsilon^{ik}{}_j \delta P^j \text{ for } Y^A = \epsilon^{AB} \nabla_B \tilde{X}^k,$$

*where $\delta P^j$ is the total flux of Bondi–Sachs linear momentum.*

In particular, $\delta J$ is invariant under pure supertranslation and the transformation law is the same as the special-relativistic angular momentum (1.1). The assumption on $N_{AB}$ is needed even for the convergence of $\delta J(Y)$. The limit of the quasilocal center of mass integral defined in [8, 9] also gives a new definition of center of mass integral at null infinity [25, 12]. The center of mass integral $C(u, Y)$ of a $u$ cut and with respect to a boost BMS field $Y = \nabla^a \tilde{X}^k \dfrac{\partial}{\partial x^a} + u \tilde{X}^k \dfrac{\partial}{\partial u}$ is defined to be:

$$\begin{aligned} C(u,Y) = & \int_{S^2} \nabla^A \tilde{X}^k \left( N_A - \frac{1}{4} C_{AB} \nabla_D C^{DB} - \frac{1}{16} \nabla_A \left( C^{DE} C_{DE} \right) \right) - 2u \int_{S^2} (\tilde{X}^k m) \\ & + \int_{S^2} c \left( 3 \tilde{X}^k m - \nabla^A \tilde{X}^k \nabla_A m \right) \\ & + \int_{S^2} \left( 2 \nabla^A \tilde{X}^k \epsilon_{AB} (\nabla^B \underline{c}) m - \frac{1}{16} \tilde{X}^k \nabla_A (\Delta + 2) \underline{c} \nabla^A (\Delta + 2) \underline{c} \right). \end{aligned} \tag{4.3}$$

## 5 Concluding remarks

We obtain a complete set of ten conserved quantities $(E, P^k, J^k, C^k)$ at null infinity (all as functions of the retarded time $u$ and can be calculated numerically in a straightforward way) that satisfy the following properties:

- $(E, P^k, J^k, C^k)$ all vanish for any Bondi–Sachs coordinate system of the Minkowski spacetime where there is no gravity.
- In a Bondi–Sachs coordinate system of the Kerr spacetime, $P^k$ and $C^k$ vanish, and $E$ and $J^k$ recover the mass and angular momentum.
- On a general spacetime, the total fluxes of $(E, P^k, J^k, C^k)$ are supertranslation invariant.
- $(E, P^k, J^k, C^k)$ and their fluxes transform according to basic physical laws under ordinary translations (1.1).

Since these are independent of the choice of the Bondi–Sachs coordinate, they can be computed in an arbitrary one. In addition, as the definition comes from quasilocal formulation, one is able to calculate it at a finite distance. It would be of great interest to explore more detailed calculation of these conserved quantities for data coming from observations.

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