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Initial conditions for the inflation with flat potential

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We discuss the naturality of the standard chaotic initial conditions in inflationary models with flat scalar potential, which are favorable by the present cosmological data.

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1 Inflationary solution to the Hot Big Bang problems

The Standard Model of particle physics (SM) and General Relativity (GR) fail to describe the cosmology properly. The major phenomenological problems are dark matter phenomena and the baryon (matter–antimatter) asymmetry of the Universe. There are several puzzling coincidences between cosmological parameters, among them the widely recognized is an order-of-magnitude equality of baryon, dark matter and dark energy contributions to the present energy density of the Universe. The vanilla ΛCDM model seems to exhibit lack of dwarf galaxies and other issues probably related to evolution of the matter perturbations at small scales.

There is a separate class of problems called *the problem of initial conditions for the Hot Big Bang Theory.* Since the homogeneous Universe may form spherical or hyperbolic 3-dimensional manifold, its flatness at present is mysterious. Both at radiation and at matter dominated stages of the Universe expansion the size of the causally connected region grows faster than the physical size does, so the Universe must be highly inhomogeneous at large scales, that is quite contrary to all observations. The Universe expansion is adiabatic, keeps the entropy conserved in a comoving volume, hence the entropy density in the very early Universe was enormously high $s \gg M_{\rm Pl}^3$. Both radiation and matter dominated stages imply initial singular energy density at the beginning of expansion, $\rho \propto 1/t^2$. If SM is a remnant of some Grand Unified Theory, the corresponding phase transition(s) at the Hot stage generally populates the Universe with topologically defects; later the monopoles or domain walls become dominant and overclose the Universe.

The initial condition problems can be elegantly solved by introducing an (exponentially) fast expansion preceding the Hot stage. Then the curvature (if any) grows

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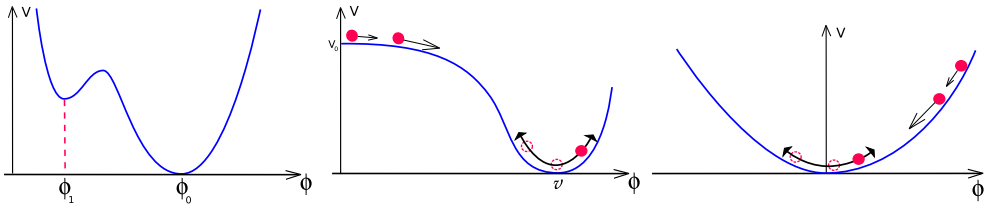


Figure 1 Scalar potentials $V(\phi)$ of "old inflation" (left panel), "new inflation" (middle panel) and "chaotic inflation" (right panel).

exponentially and hence the Universe is exponentially flat. An originally causally connected region however tiny becomes exponentially large and today we observe nothing but its minuscule part. Any heavy stable relics (like monopoles) are at exponentially large distances one from each other, so the late Hot Universe is safe. There are no initial singularities in the exponentially expanding Universe.

This stage, dubbed *inflation*, can be naturally organized by a slowly evolving homogeneous scalar field ϕ, *inflaton*. In the "old inflation" [1] the scalar field settles in a false vacuum (see the left panel in Fig. 1) which provides with exactly exponential expansion terminated by tunneling to the true vacuum. The latter process is not effective and other variants had been put forward. In the "new inflation" [2, 3] the scalar field slowly rolls down from the hilltop of the scalar potential (see the middle panel in Fig. 1) that yields almost exponential expansion. In both models above the special starting positions of the scalar field may be attributed to a preceding Hot stage. Another idea – chaotic initial conditions – is utilized in models called *large field inflation* [4] (see the right panel in Fig. 1).

2 Chaotic inflation: unlikeness of models with flat potential

In the chaotic inflation field ϕ rolls down along the potential $V(\phi)$ starting from large, generally transplanckian values, $\phi > M_{\rm Pl}$. In the Robertson–Walker metric, $ds^2 = dt^2 - a^2(t)\,\mathbf{dx}^2$, with $a(t)$ being the scale factor, $\phi(t, \mathbf{x})$ solves equation

$$\ddot{\phi} - \frac{\Delta\phi}{a^2} + 3\,\mathcal{H}\dot{\phi} + V'(\phi) = 0\,. \tag{1}$$

The Hubble parameter $\mathcal{H} \equiv \dot{a}/a$ obeys the Friedmann equation

$$\mathcal{H}^2 = \frac{8\pi}{M_{\rm Pl}}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\nabla\phi)^2 + V(\phi)\right) \tag{2}$$

and plays a role of dynamical friction in Eq. (1). Actually, if the field evolves very slowly, its energy density is dominated by the scalar potential and the Hubble parameter remains almost constant, see Eq. (2), ensuring the almost exponential expansion. Then the gradient terms in Eqs. (1), (2), being suppressed by the scale factor $a(t) \propto \exp(\mathcal{H}t)$, get nullified and the Hubble friction governs the field homogeneous

slow rolling towards the origin of scalar potential, $\phi = \phi(t) \to 0$. The kinetic terms in Eqs. (1), (2) are negligible provided the so-called slow roll parameters are small,

$$\frac{M_{\rm Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \equiv \epsilon \ll 1\,, \quad \frac{M_{\rm Pl}^2}{8\pi}\frac{V''}{V} \equiv \eta \ll 1\,. \tag{3}$$

One can check that the latter conditions are satisfied for any power-law scalar potential, $V(\phi) \propto \phi^\alpha$, if the scalar field takes a transplanckian value, $\phi_i > M_{\rm Pl}$. Then the field goes down slowly and once $\epsilon, \eta \simeq 1$, the inflation terminates.

The chaotic initial conditions for the inflaton field imply [4, 5] that the scalar field fluctuates in any Planck-size region, and all the contributions to the energy density – the kinetic term, the gradient term and the potential term – are (on average) of the same order, which is Planckian, i.e.

$$\frac{1}{2}\dot\phi^2 \sim \frac{1}{2}(\partial_i\phi)^2 \sim V(\phi) \sim M_{\rm Pl}^4\,. \tag{4}$$

Then if in a particular Planck-size volume the potential term becomes somewhat dominant as result of the fluctuations, the dynamics of the scalar field rapidly approaches the slow roll and the chaotic inflation begins in this region.

Inflation itself is almost untestable: the only exception is the prediction of an exponentially flat Universe, that is in agreement with observations at present. However, it is a general prediction inherent in the very paradigm of inflation. To recognize a particular inflationary model we need something else. Quite remarkably, the inflation provides with an appropriate feature. Namely, inflaton inhomogeneities $\delta\phi(t, \mathbf{x})$ (linear perturbations about the homogeneous inflationary solution) originated from the inflaton quantum fluctuations can seed the matter inhomogeneities in the late Universe. Their subsequent evolution initiates large scale structure formation and explain the anisotropy of cosmic microwave background radiation $\delta T/T \sim 10^{-4}$.

The inflaton perturbations solve Eq. (1) and behave in a very peculiar way: the amplitude drops inversely with the scale factor till the very moment when the physical 3-momentum of the mode $|\mathbf{p}| \propto 1/a$ matches the Hubble parameter, $p \simeq \mathcal{H}$, and then the mode amplitude remains constant, $\delta\phi(t, \mathbf{p})$, while the stretching proceeds. Hence the amplitude of the modes is almost momentum-independent, the small decrease in the Hubble parameter during inflation results in a small tilt $|n_s - 1| \ll 1$ of the otherwise flat scalar perturbation spectrum. Both the amplitude and the tilt depend on the inflationary model parameters, and analysis of associated with matter inhomogeneities cosmological observables allows us to discriminate between various inflationary models, for details see e.g. [6]. Figure 2 shows that the present data tend to favor the large field inflation models with *concave* scalar potential. Among them are theoretically well motivated models with asymptotically flat potentials: R^2-inflation [7], Higgs-inflation [8], moduli inflation in supersymmetric extensions of the SM, etc. These models seem to be unlikely in the framework of the chaotic inflation [9]. Indeed, at the Planck epoch, when kinetic and gradient terms take the Planckian values according to conditions (4), the potential (being flat) is fixed by the cosmological observations at much smaller value,

$$\frac{1}{2}\dot\phi^2 \sim \frac{1}{2}(\partial_i\phi)^2 \sim M_{\rm Pl}^4 \gg V(\phi)\,, \tag{5}$$

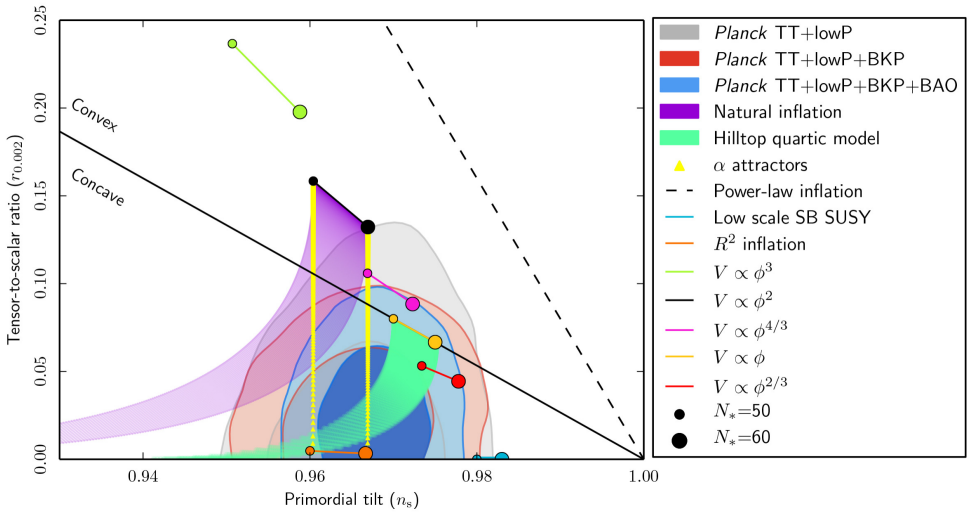


Figure 2 Cosmological constraints [10] on inflationary models. The colored regions are allowed at 95% CL, there are predictions of various cosmological models.

so that slow roll never happens. In such a model the inflation requires a beginning with all terms of the same order that is much smaller than the Planckian value, which seems unnatural. Below we illustrate the problem with two prominent examples and resolve it by explaining the proper way to implement the chaotic initial conditions, for more details see [11].

3 An example: R^2-inflation

Let us consider the first well-developed inflationary model [7], based on modified gravity with action

$$S^{JF} = -\frac{M_P^2}{2}\int\sqrt{-g}\,d^4x\left(R - \frac{R^2}{6\,\mu^2}\right),$$

which after the convenient Weyl transformation (from the Jordan frame to the Einstein frame),

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \chi\, g_{\mu\nu}\,, \qquad \chi = \exp\left(\sqrt{2/3}\,\phi/M_P\right),$$

casts into the standard Einstein–Hilbert term and terms describing the scalar degree of freedom, dubbed *scalaron*, emerging in the gravity sector,

$$S^{EF} = \int\sqrt{-\tilde{g}}\,d^4x\left[-\frac{M_P^2}{2}\,\tilde{R} + \frac{1}{2}\,\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{3\,\mu^2 M_P^2}{4}\left(1 - \frac{1}{\chi(\phi)}\right)^2\right].$$

The model has an exponentially flat potential, which approaches the value

$$V_0 \simeq 10^{-12}\,M_{\rm Pl}^4$$

fixed by matching the amplitude of scalar perturbations to $\sim 10^{-4}$. Naively, within the chaotic paradigm the model suffers from unlikeness problem (5).

This is not true in fact. Indeed, in the Jordan frame the chaotic initial conditions look like $|R^{JF}| \sim M_{\rm Pl}^2$, and matching to cosmological data infers

$$\chi^2 = 1 - \frac{R^{JF}}{3\,\mu^2} \sim 10^{12}\,.$$

The scalar curvature in the Jordan frame is related to that in the Einstein frame and scalaron as follows,

$$-R^{JF} = -\chi^2\,R^{EF} + \chi^2\,\frac{8\pi}{M_{\rm Pl}^2}\,g^{EF}_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - \chi^2\,\frac{4\sqrt{3\pi}}{M_{\rm Pl}}g^{EF}_{\mu\nu}\partial^\mu\partial^\nu\phi\,.$$

The first and second terms in the r.h.s. above refer to the pure gravity and scalaron contributions to the energy density, respectively. Hence they must be of the same order within the paradigm of chaotic inflation. The third term happens to be of the same order provided the equations of motion.

Therefore we have

$$\left|R^{JF}\right|\chi^{-2} \sim \left|R^{EF}\right| \sim \dot\phi^2 M_{\rm Pl}^2 \sim (\partial_i\phi)^2 M_{\rm Pl}^2\,,$$

which means that all the terms in the Einstein frame are actually of the same order, consistent with the requirement of successful inflation,

$$\frac{1}{2}\dot\phi^2 \sim \frac{1}{2}(\partial_i\phi)^2 \sim V(\phi) \sim 10^{-12}\times M_{\rm Pl}^4\,.$$

There is no unlikeness problem in the R^2-inflation.

4 One more example: Higgs inflation

To cook the successful inflation with the SM Higgs doublet H we must introduce its non-minimal coupling to gravity [8],

$$S = \int d^4x\sqrt{-g}\left(-\frac{M_P^2}{2}R - \xi H^\dagger H R + \mathcal{L}_{SM}\right)$$

or written in the unitary gauge $H^T = \left(0\,,(h+v)/\sqrt{2}\right)$ (neglecting $v = 246\,\mathrm{GeV}$)

$$S = \int d^4x\sqrt{-g}\left(-\frac{M_P^2+\xi\,h^2}{2}R + \frac{(\partial_\mu h)^2}{2} - \frac{\lambda\,h^4}{4}\right).$$

Performing the Weyl transformation

$$g_{\mu\nu} = \Omega^{-2}\tilde g_{\mu\nu}\,,\qquad \Omega^2 = 1 + \frac{\xi\,h^2}{M_P^2}$$

one finds in the Einstein frame for the canonically normalized field χ

$$\frac{d\chi}{dh} = \frac{M_P\sqrt{M_P^2 + (6\xi+1)\xi\, h^2}}{M_P^2 + \xi\, h^2}$$

the scalar potential

$$U(\chi) = \frac{\lambda M_P^4\, h^4(\chi)}{4(M_P^2 + \xi\, h^2(\chi))^2} = \frac{\lambda M_P^4}{4\xi^2}\left(1 - \exp\left(-\frac{\sqrt{2}\,\chi}{\sqrt{3}M_P}\right)\right)^2$$

at relevant for inflation values $h \gg M_P/\sqrt{\xi}$. It looks exactly as the scalaron potential in the previous example of R^2-inflation, so we naively expect here the same unlikeness problem.

Actually, there is no unlikeness problem here as well. Similar to the case of R^2-inflation, one starts with the chaotic initial conditions implemented in the Jordan frame,

$$\Omega^2\, M_{\rm Pl}^2\, R^{JF} \sim \xi\dot{h}^2 \sim \xi(\partial_i h)^2 \sim \lambda h^4 \sim M_{\rm Pl}^4\,.$$

Here we account for the fact that the nonminimal gravity coupling to the Higgs field effectively changes the Higgs kinetic and gradient terms, and chose the dominant parts. Consequently,

$$R^{JF} \sim M_{\rm Pl}^2/\Omega^2$$

and cosmological observations fix $V_0^{EF} \sim 10^{-12} \times M_{\rm Pl}^4$, hence

$$\Omega^2 \sim 10^6\,.$$

Then in the Einstein frame one arrives at the same picture: all the terms are of the same order needed for the successful inflation, namely

$$\frac{1}{2}\dot{\phi}^2 \sim \frac{1}{2}(\partial_i\phi)^2 \sim M_{\rm Pl}^4/\Omega^4 \sim 10^{-12} \times M_{\rm Pl}^4\,.$$

There is no unlikeness problem in the Higgs-inflation either.

Conclusion: initial conditions in an arbitrary frame

We find that Chaotic paradigm works perfectly well for both R^2-inflation and Higgs-inflation: they do not suffer from the unlikeness problem. The presented analysis suggests a simple prescription for implementing the chaotic initial conditions in any model as follows. Start in any frame with all contributions to the energy density being of the same order (say, $\sim M_{\rm Pl}^4$ or any other value). Then in any other frame the corresponding contributions are always of the same (though another) order (say, $\sim 10^{-12} \times M_{\rm Pl}^4$ in the EF).

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