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Instability in classical and modified gravity

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Gravitational instability in classical Jeans theory, General Relativity, and modified gravity is considered. A simple suggestion to eliminate the problem with zero order approximation in the Jeans theory is proposed. It is shown that density perturbations in the background with an increasing density rise faster than in the standard case of time independent background. Evolution of density and metric perturbations in the background of high frequency oscillations of curvature in $F(R)$ gravity is studied. In addition to the usual Jeans-like instability new effects of amplification of perturbations, associated with parametric resonance and antifriction phenomena, are found.

Keywords: Jeans theory, General Relativity, modified gravity

1 Introduction

The instability of self-gravitating systems was first investigated by Jeans [1] in non-relativistic Newtonian gravity. It was extended to General Relativity (GR) by Lifshitz [2] and nowadays it is widely used in cosmology to study the rise of density perturbations in the expanding universe [3–6]. The original Jeans approach is based on the Poisson equation

$$\Delta\Phi = 4\pi G\varrho\,, \tag{1.1}$$

which is not satisfied in zeroth order approximation, because the potential Φ is considered as a first order quantity, while the matter density $\varrho = \varrho_{\rm b} + \delta\varrho$ includes zero (background) and first order terms. To cure this shortcoming, Mukhanov [5] suggested adding an antigravitating substance, such as e.g. vacuum-like energy, which would counterbalance the gravitational attraction of the background, so that Eq. (1.1) would be satisfied at zeroth order. Alternatively, in [7] the authors assumed that the background density is zero, so Eq. (1.1) becomes a relation between first order terms. This problem is absent in cosmology, where the zeroth order background equations

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are satisfied. They are the usual Friedmann equations in a homogeneous, isotropic universe, see the books [3–6].

In our paper [8] we take a different approach to the classical Jeans problem, assuming that Eq. (1.1) is valid for zeroth order terms so the solution of the equations of motion leads to time dependent background energy density and gravitational potential. The characteristic time of variation of these quantities is close to the Jeans time [both are essentially the gravitational time $t_{\rm g} \sim (G\varrho)^{-1/2}$], so the development of the Jeans instability goes faster than in the standard theory, where such effect is not taken into account.

In [8], we studied the Jeans instability in classical and modified gravity in a background with rising energy density, assuming that the background metric changes slowly as a function of space and time compared to the typical frequency and Compton wavelength of the scalaron field. In [9], we investigate gravitational instability in a quickly oscillating curvature background. The time evolution of first-order perturbations is governed by a fourth-order differential equation (instead of the usual second order one), therefore new types of unstable solutions are expected, and have in fact been found.

There appear not only the usual Jeans solution with a slightly modified (reduced) length scale, but also parametric resonance amplification of density perturbations, and an amplification of the perturbations due to the "antifriction" behaviour of the coefficients of odd (first and third) derivatives in the equation.

2 Jeans instability in Newtonian theory with space and time dependent background

We consider a spherically symmetric cloud of particles with initially vanishing pressure and velocities, and study the classical non-relativistic Jeans problem in Newtonian gravity. The essential equations are the well known Poisson, Euler, and continuity equations:

$$\Delta\Phi = 4\pi G\varrho, \tag{2.1}$$

$$\partial_{\rm t}(\varrho\mathbf{v}) + \varrho(\mathbf{v}\nabla)\mathbf{v} + \nabla P + \varrho\nabla\Phi = 0, \tag{2.2}$$

$$\partial_{\rm t}\varrho + \nabla(\varrho\mathbf{v}) = 0. \tag{2.3}$$

We consider the time dependent problem taking as initial value a homogeneous distribution $\varrho = const$ inside a sphere with radius $r_{\rm m}$, while outside this sphere $\varrho = 0$. The initial values of particle velocities and pressure are taken to be zero and the potential Φ is supposed to be a solution of the Poisson equation (2.1):

$$\Phi_0(r > r_{\rm m}) = -MG/r, \;\; \Phi_0(r < r_{\rm m}) = 2\pi G\varrho_0 r^2/3 + C_0, \tag{2.4}$$

where $C_0 = -2\pi G\varrho_0 r_{\rm m}^2$ to make the potential continuous (the value of C_0 is not important for us) and the total mass of the gravitating sphere is $M = 4\pi\varrho_0 r_{\rm m}^3/3$.

In what follows we will be interested in the internal solution for $r < r_{\rm m}$. Now we can find how the background quantities ϱ, v, and P evolve with time at small t. From Eq. (2.2) it follows that:

$$v_1(r,t) = -\nabla\Phi_0 t = -4\pi G\varrho_0 rt/3. \tag{2.5}$$

From the continuity equation we find

$$\varrho_1 = \frac{2\pi}{3} G \varrho_0^2 t^2 \ \text{ or } \ \varrho_{\rm b}(t,r) = \varrho_0 + \varrho_1 = \varrho_0 \left(1 + \frac{2\pi}{3} G \varrho_0 t^2\right). \tag{2.6}$$

It is interesting that ϱ rises with time in such a way that it remains constant in space. Because of the homogeneity of ϱ the pressure remains zero, i.e. $P_1 = 0$.

The time variation of the background potential is found using (2.1):

$$\Phi_{\rm b}(r,t) = \Phi_0 + \Phi_1 = \frac{2\pi}{3} G r^2 \varrho_0 \left(1 + \frac{2\pi}{3} G \varrho_0 t^2\right). \tag{2.7}$$

Now we can study evolution of perturbations over this time-dependent background. We proceed as usual, writing $\varrho = \varrho_{\rm b}(r,t) + \delta\varrho$, $\Phi = \Phi_{\rm b}(r,t) + \delta\Phi$, $v = v_1(r,t) + \delta v$, and $\delta P = c_s^2 \delta\varrho$, where c_s is the speed of sound. Here all δ-quantities are infinitesimal and are neglected beyond first order. In what follows we also neglect the products of small sub-one quantities (i.e. ϱ_1 etc) with δ's. This significantly simplifies the calculations, while of course the results do not change significantly. We find:

$$\Delta(\delta\Phi) = 4\pi G \delta\varrho, \tag{2.8}$$

$$\partial_t \delta\mathbf{v} + \nabla\delta\Phi + \delta\varrho/\varrho_0 \nabla\Phi_{\rm b} + \nabla\delta P/\varrho_0 = 0, \tag{2.9}$$

$$\partial_t \delta\varrho + \varrho_0 \nabla(\delta\mathbf{v}) = 0. \tag{2.10}$$

Making the Fourier transform of Eqs. (2.8)–(4.11) and neglecting the r-dependent term $(\delta\varrho/\varrho_0)\nabla\Phi_{\rm b}$, we obtain the eigenvalue equation:

$$k^2(\gamma^2 - c_s^2 k^2 + 4\pi G \varrho_0) = 0\,. \tag{2.11}$$

For small k we find the usual exponential Jeans instability:

$$\frac{\delta\varrho_{J1}}{\varrho_0} \sim \exp\left[t\left(4\pi G\varrho_0 - c_s^2 k^2\right)^{1/2}\right]. \tag{2.12}$$

However, these small perturbations have the same characteristic rising time $\sim 1/(4\pi G\varrho_0)^{1/2}$ as that of the classical rise of ϱ_1. We can estimate the impact of the rising background energy density on the rise of perturbations making an adiabatic approximation, that is replacing the exponent in Eq. (2.12) with the integral:

$$\frac{\delta\varrho_{J2}}{\varrho_0} \sim \exp\left\{\int_0^t dt\left[4\pi G\varrho_{\rm b}(t,r) - k^2 c_s^2\right]^{1/2}\right\}. \tag{2.13}$$

where $\varrho_{\rm b}(t,r)$ is given by Eq. (2.6).

Estimating the above integral for small k we find that the enhancement factor $\delta\varrho_{J2}/\delta\varrho_{J1}$ is equal to 1.027 after a time $t = t_{grav}$, where $t_{grav} = 1/\sqrt{4\pi G\varrho_0}$, while for $t = 2\,t_{grav}$ it is 1.23, for $t = 3\,t_{grav}$ it is 1.89, and for $t = 5\,t_{grav}$ it is 11.9. Note that to derive (2.12) and (2.13) we assumed that $t < t_{grav}$, so we should not treat these factors as numerically accurate; still, we can interpret them as an indication that the rise of fluctuations is indeed faster than in the usual Jeans scenario.

3 Density perturbations in General Relativity

In this section we consider the evolution of scalar perturbations in General Relativity (GR). The treatment of the Jeans instability in GR starts from the Einstein equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 8\pi G\, T_{\mu\nu} \equiv \widetilde{T}_{\mu\nu}\,, \tag{3.1}$$

which automatically include the equations of motion of matter, namely the continuity and Euler equations.

As we did previously, we consider a spherically symmetric cloud of matter with initially constant energy density inside the limit radius $r = r_m$. We choose Schwarzschild-like isotropic coordinates in which the metric takes the form:

$$ds^2 = A dt^2 - B\, \delta_{ij}\, dx^i dx^j\,, \tag{3.2}$$

where the functions A and B may depend upon r and t.

The energy-momentum tensor is taken in the perfect fluid form without dissipative corrections and has the following components:

$$T_{tt} = (\varrho + P) U_t^2 - PA \approx \varrho A\,, \tag{3.3}$$

$$T_{jt} = (\varrho + P) U_t U_j \approx -(\varrho + P) B v_j\,, \tag{3.4}$$

$$T_{ij} = (\varrho + P) U_i U_j - P g_{ij} \approx PB\delta_{ij}\,. \tag{3.5}$$

We assume that the background metric changes slowly as a function of space and time and study small fluctuations around the background quantities: $\varrho = \varrho_{\rm b} + \delta\varrho$, $\delta P = c_s^2 \delta\varrho$, $\mathbf{v} = \delta\mathbf{v}$, $A = A_{\rm b} + \delta A$, $B = B_{\rm b} + \delta B$.

The corresponding linear equations for the infinitesimal perturbations are:

$$-\Delta \delta B = \delta\widetilde{\varrho}\,, \tag{3.6}$$

$$\partial_i \partial_j (\delta A + \delta B) = 0\,. \tag{3.7}$$

$$\delta\dot{\varrho} + \varrho_{\rm b}\, \partial_j \delta v^j + \frac{3}{2}\varrho_{\rm b}\, \delta\dot{B} = 0\,, \tag{3.8}$$

$$\varrho\, \delta\dot{v}_j + \partial_j \delta P + \frac{1}{2}\varrho_{\rm b}\, \partial_j \delta A = 0\,. \tag{3.9}$$

We look for the solution in the form $\sim \exp[-i\gamma t + i\mathbf{k}\cdot\mathbf{x}]$ and obtain the following expressions for the frequency eigenvalues:

$$\gamma^2 = \frac{c_s^2 k^2 - \widetilde{\varrho}_{\rm b}/2}{1 + 3\widetilde{\varrho}_{\rm b}/(2k^2)}\,, \tag{3.10}$$

where $\widetilde{\varrho}_{\rm b} = 8\pi G \varrho_{\rm b}$. This result almost coincides with the Newtonian one (4.11). An extra term in the denominator is induced by the volume variation, and it is small for $k \sim k_J = \sqrt{4\pi G \varrho_0}/c_s$.

4 Density perturbations over a quickly varying background

Gravity modifications at large distances have been suggested [10] for an explanation of the accelerated cosmological expansion observed in the present-day universe. The idea is to add a non-linear function of curvature scalar, $F(R)$, to the standard Einstein-Hilbert action:

$$A_{\rm grav} = -\frac{m_{Pl}^2}{16\pi}\int d^4x\sqrt{-g}\,\left[R + F(R)\right]\,, \tag{4.1}$$

such that the modified Einstein equations have an accelerated De Sitter-like solution in the absence of matter. Some examples of $F(R)$-theories which realistically describe the cosmological acceleration can be found in [11–14].

The modified Einstein field equations, derived from the action (4.1), are:

$$(1+F'_R)\,R_{\mu\nu} - \frac{1}{2}\,(R+F)\,g_{\mu\nu} + (g_{\mu\nu}D_\alpha D^\alpha - D_\mu D_\nu)\,F'_R = \widetilde{T}_{\mu\nu}, \tag{4.2}$$

where $F'_R = dF/dR$ and D_μ is the covariant derivative.

We assume that the background metric weakly deviates from the Minkowski one, while derivatives of the metric may be far from their GR values. In particular, R may be very much different from $R_{\rm GR} = -\widetilde{T}$. We consider astronomical systems with $|R_c| \ll |R| \ll m^2$, where R_c is the present cosmological curvature and m is the scalaron mass. Both limits are natural for relatively dense systems, that is denser than the average cosmological background but much less dense than $m_{Pl}^2 m^2$. Note that $m \gtrsim 10^5$ GeV, see e.g. [15]. In this limit we have

$$|F(R)| \ll |R|\,,\; |F'(R)| \ll 1\,. \tag{4.3}$$

Therefore, Eq. (4.2) can be approximated as:

$$G_{\mu\nu} + \frac{1}{3\omega^2}(D_\mu D_\nu - g_{\mu\nu}D^2)R = \widetilde{T}_{\mu\nu}\,, \tag{4.4}$$

where $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$ is the Einstein tensor and

$$\omega^{-2} = -3F''_{RR}\,. \tag{4.5}$$

Once written in this form, the equation is largely independent of the specific model considered except of course of the value of ω, and provided that along the background solution $\omega \simeq$ const., or rather $\dot{\omega}/\omega^2 \ll 1$, and that the conditions (4.3) are fulfilled.

As usual the metric and the curvature tensors are expanded around their background values at first order in infinitesimal perturbations, i.e.

$$A = A_{\rm b} + \delta A\,,\quad B = B_{\rm b} + \delta B\,,\quad R = R_{\rm b} + \delta R\,, \tag{4.6}$$

where $A_{\rm b}$ and $R_{\rm b}$ are quickly oscillating functions of time with possibly large amplitude ("spikes") which were found in our previous works [16, 17] and $A_{\rm b} = 1 + R_{\rm b}r^2/6$.

Using the technique described in [8, 9], we derive the fourth order equation for the single function δB which in the limit of small kr has the form:

$$\delta\dddot{\dddot{B}} - \left(1+\frac{2k^2}{3\omega^2}\right)\frac{\dot{R}_b}{2k^2}\,\delta\ddot{\ddot{B}} + \\ + \left[\omega^2 - \frac{\widetilde{\varrho}_b\omega^2}{2k^2}\left(1+\frac{8k^2}{3\omega^2}\right) + k^2(1+c_s^2) - \frac{\ddot{R}_b}{k^2}\left(1+\frac{2k^2}{3\omega^2}\right)\right]\delta\ddot{\dddot{B}} + \\ + \left[-\frac{\dddot{R}_b}{2k^2}\left(1+\frac{2k^2}{3\omega^2}\right) - \frac{\dot{R}_b c_s^2}{2}\left(1+\frac{2k^2}{3\omega^2}\right)\right]\delta\dot{\dddot{B}} + \\ + \left[\frac{\dot{R}_b r^2}{12}\left(\omega^2 + k^2(1-c_s^2) + \frac{2\widetilde{\varrho}_b}{3} - \frac{\widetilde{\varrho}_b\omega^2}{2k^2}\right)\right]\delta\dot{\ddot{B}} + \\ + \left[c_s^2k^2(k^2+\omega^2) - 2c_s^2\widetilde{\varrho}_b\omega^2\left(1+\frac{2k^2}{3\omega^2}\right) - \frac{\widetilde{\varrho}_b\omega^2}{2}\left(1+\frac{4k^2}{3\omega^2}\right)\right]\delta\ddot{B} = 0\,. \quad (4.7)$$

We introduce the dimensionless time $\tau = \omega t$, and define:

$$\delta B \equiv z\,, \quad R_{\rm b} = -\widetilde{\varrho}_{\rm b} y\,, \quad a = \frac{\widetilde{\varrho}_{\rm b}}{k^2}\,, \quad b = \frac{k^2}{\omega^2}\,, \quad c = c_s^2\,. \quad (4.8)$$

We can further simplify the notation by introducing the following parameters:

$$\alpha = \frac{a}{2}\left(1+\frac{2b}{3}\right)\,, \quad (4.9a)$$

$$\Omega^2 = 1 - \frac{a}{2}\left(1+\frac{8b}{3}\right) + b(1+c)\,, \quad (4.9b)$$

$$\mu = b\left[c(1+b) - \frac{a}{2}\left(1+\frac{4b}{3}\right) - 2ac\left(1+\frac{2b}{3}\right)\right]. \quad (4.9c)$$

In terms of these quantities, Eq. (4.7) takes a form convenient for qualitative and quantitative analysis:

$$z'''' + \alpha y' z''' + (\Omega^2 + 2\alpha y'')z'' + \alpha(y''' + bcy')z' + \mu z = 0\,. \quad (4.10)$$

Since the physically interesting quantity is the magnitude of density perturbations, we present $\delta\varrho/\varrho_{\rm b}$ expressed through $z \equiv \delta B$:

$$\frac{\delta\varrho}{\varrho_{\rm b}} = z\left[\frac{1+b}{a(1+2b/3)} - 2\right] + \frac{1}{2}\,z'y' + \frac{z''}{a(1+2b/3)}\,. \quad (4.11)$$

When k is close to its Jeans value, $k \sim k_J$, $a = \widetilde{\varrho}_{\rm b}/k^2 \sim c_s^2$ and the first term in the square brackets dominates if $c_s^2 < 1/2$, which is surely the case for non-relativistic systems.

4.1 Modified Jeans instability

Let us compare our equation (4.10) with the usual Jeans equation

$$\delta\ddot{\varrho} + \left(c_s^2k^2 - \frac{\widetilde{\varrho}_{\rm b}}{2}\right)\delta\varrho = 0\,. \quad (4.12)$$

As is well known, the sign of the term $c_s^2k^2 - \widetilde{\varrho}_{\rm b}/2$ determines the stability of solutions, which are sound waves for $c_s^2k^2 > \widetilde{\varrho}_{\rm b}/2$ and unstable modes in the opposite regime. These unstable modes appear at the Jeans scale

$$k^2 \leq (k_J^{\rm GR})^2 \equiv \frac{\widetilde{\varrho}_{\rm b}}{2c_s^2}\,. \tag{4.13}$$

In modified gravity the condition of stability is determined by the sign of μ (4.9c). In the limit of small amplitude of curvature oscillations or very low frequency, we can neglect $y(\tau)$ in Eq. (4.10) so that it is reduced to a simple equation with constant coefficients which is solved by the substitution $z = \exp(\gamma\tau)$. The eigenvalue γ is thus determined by the algebraic equation:

$$\gamma^4 + \Omega^2\gamma^2 + \mu = 0\,, \tag{4.14}$$

where Ω^2 is given by Eq. (4.9b). The eigenvalues γ^2 solving this are:

$$\gamma^2 = -\frac{\Omega^2}{2} \pm \sqrt{\frac{\Omega^4}{4} - \mu}. \tag{4.15}$$

If $\mu < 0$, then one of the roots $\gamma^2 > 0$, so one of the eigenvalues is positive. It corresponds to the usual exponential Jeans instability, though the values of the Jeans wave vector in modified gravity and in GR are different. We present an explicit solution for large $\omega^2 \gg k^2, \widetilde{\varrho}$:

$$(k_J^{\rm MG})^2 = (k_J^{\rm GR})^2\left[1 + \frac{(k_J^{\rm GR})^2}{3\omega^2}\right], \tag{4.16}$$

which recovers the GR result in the limit $\omega \to \infty$.

Equation (4.16) shows that in modified gravity the Jeans wavenumber is larger than in General Relativity. This corresponds to a reduced minimum length scale associated to structure formation. The correction is typically small, but for models in which $k_J^{\rm GR}/\omega$ is non negligible it could lead to significant corrections. The phenomenology of this result and its implications to constrain models will be dealt with elsewhere.

If μ is positive, but $\mu < \Omega^4/4$, both possible values of γ^2 are real and negative, so γ is purely imaginary which corresponds to acoustic oscillations. Thus these two cases of negative and positive (but not too large, see below) μ are in a one-to-one correspondence to the usual Jeans analysis.

If μ is large, i.e. $\mu \geq \Omega^4/4$, there would exist a new type of unstable oscillating solution with exponentially rising amplitude. Indeed, γ^2 becomes a complex number and two of the four solutions of Eq. (4.14) for γ have positive real parts. However, such solution is absent in the model studied here. On the other hand, it is not excluded that some other modified gravity models may possess such a kind of instability.

If $y(\tau)$ is non-negligible, quite interesting new effects can show up. The function $y(\tau)$ entering the fourth order dimensionless equation (4.10) is an oscillating function of "time" τ. It can induce an analogue of the parametric resonance instability resulting in a very fast rise of perturbations at a certain set of frequencies. Another new effect can be called "antifriction". It appears at sufficiently large amplitudes

of oscillations of $y(\tau)$ such that the coefficients in front of the odd derivative terms become periodically negative. This phenomenon leads to an explosive rise of z in a wide range of frequencies. Both effects do not exist in the standard General Relativity and, if discovered, would be a proof of modified gravity. On the contrary, the non-observation of these effects would allow us to put stringent restrictions on the parameters of $F(R)$-theories.

4.2 Parametric resonance

In this section we show that even for $\mu > 0$, corresponding classically to the stable (sound-wave) regime, Eq. (4.10) may admit unstable solutions, through a mechanism analogous to parametric resonance.

The usual textbook example of parametric resonance is the Mathieu equation (for details, see e.g. [18]):

$$\ddot{f}(\tau) + \Omega_0^2 \left[1 + h\,\cos(\Omega_1 + \epsilon)\tau\right] f(\tau) = 0\,, \tag{4.17}$$

where it is assumed that $h \ll 1$. If $\Omega_1/\Omega_0 = 2$, the solution grows exponentially, behaving approximately as:

$$f(\tau) \sim \sin(\Omega_0\tau + \varphi)\,\exp(\gamma\tau)\,, \tag{4.18}$$

with $\gamma = h\,\Omega_0/4$. Parametric resonance is excited for

$$|\epsilon| < \frac{h\,\Omega_0}{2}\,. \tag{4.19}$$

There are other resonance modes, which are generated at the frequencies $\Omega_1/\Omega_0 = 2/n$, where n is an integer, but they are usually noticeably weaker.

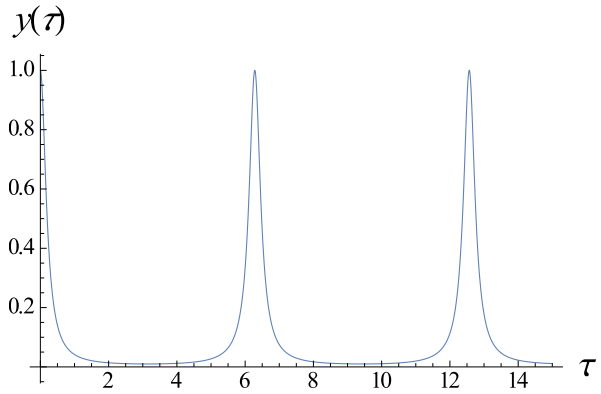


Figure 1 Spiky function $y(\tau)$ defined by Eq. (4.22) with $y_0 = 1$, $\Omega_2 = 0.5$, and $d = 0.1$.

In modified gravity, we have derived the 4th order equation (4.10) governing the evolution of scalar perturbations over an oscillating background, which enters through the function $y(\tau)$. An effect analogous to parametric resonance appears in this situation as well. A similarity of parametric resonance in the fourth-order equation

and in the classical Mathieu equation can be seen both in the limit of high and low frequencies. In the former case one can neglect lower order derivatives in Eq. (4.10), keeping only z'''', z''', and z'', effectively reducing the equation to a second order one in the variable z''. In the case of low frequencies, on the other hand, z'''' and z''' can be neglected and we come again to the Mathieu equation. In what follows we study the complete fourth order equation both analytically and numerically and observe resonant amplification.

We assume that $y(\tau)$ is a periodic function describing curvature oscillations and consider two possible forms of them:

1. purely harmonic ones:

$$y_{harm}(\tau) = y_{eq}(\tau) + y_0 \cos(\Omega_1 \tau + \theta)\,, \tag{4.20}$$

where $\Omega_1 = \omega_1/\omega$ is the dimensionless frequency, y_0 is the amplitude of oscillations and θ is a constant phase. This form of y physically corresponds to an oscillating scalar curvature [see (4.8)]:

$$R_{\rm b}(t) = -\widetilde{\varrho}_{\rm b}[y_{eq}(t) + y_0 \cos(\omega_1 t + \theta)] = R_{\rm GR}[y_{eq} + y_0 \cos(\omega_1 t + \theta)]\,. \tag{4.21}$$

where y_{eq} is the equilibrium point of the potential around which the curvature oscillates, i.e. the external energy density in units of the initial density $y_{eq} = \varrho(t)/\varrho_0$. The harmonic solution y_{harm} was found in our paper [16] when the amplitude of curvature oscillations is small. In this case $\omega_1 = \omega$ and $y_0 < y_{eq}$. The time variation of y_{eq} and y_0 is much slower than the oscillations, namely $\dot{y}_{eq}/y_{eq} \ll \omega_1$ and similarly for y_0.

2. spiky solutions found in Ref. [16], which, up to a slowly changing term, we approximate as

$$y_{sp}(\tau) = \frac{y_0\, d^2}{d^2 + \sin^2(\Omega_2 \tau + \theta)}\,, \tag{4.22}$$

where $d \ll 1$, so that we have narrow peaks with a large separation between them (see Fig. 1). $\Omega_2 \simeq 1/2$ in agreement with the solution of Ref. [16].

Let us start from the harmonic oscillations. Equation (4.10) is reduced to the Mathieu equation (4.17), if the terms containing odd derivatives in Eq. (4.10) can be neglected and $\mu \ll 1$. We can see that this is the case when α, a, b and c are all much smaller than unity. Although this seems rather restrictive, it is actually a natural possibility as it corresponds to $k \ll \omega$, $c_s^2 \ll 1$ and $k^2 \gg \widetilde{\varrho}_{\rm b}$. In this case the substitution $z'' = x$ yields:

$$x'' + \left[\Omega^2 + 2\alpha \cos(\Omega_1 \tau)\right] x = 0\,. \tag{4.23}$$

If $\Omega_1/\Omega = 2$, then the parametric resonance would be excited. Comparing this to Eq. (4.17), we see that the parameter h is expressed through α as

$$h = 2\alpha \Omega^{-2}\,. \tag{4.24}$$

We solved Eq. (4.10) numerically for different values of y_0 and $\Omega_{1,2}$. The parametric resonance excitation for harmonic oscillations is observed at the expected frequency

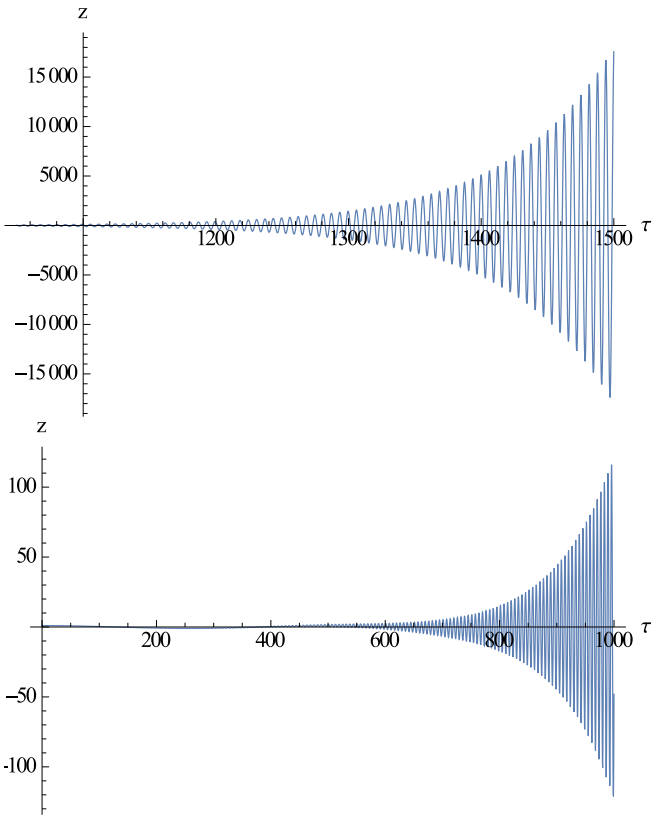


Figure 2 *Upper panel*: Parametric resonance excitation of $z(\tau)$ for harmonic curvature oscillations with $y_0 = 5$ and $\Omega_1/\Omega = 2$. *Lower panel*: Parametric resonance excitation of $z(\tau)$ for spike-like curvature oscillations with $y_0 = 30$ and $\Omega_2/\Omega = 0.5$.

$\Omega_1 = 2\Omega$, see Fig. 2 (upper panel). Here and in what follows the parameters of the medium and the wave number of the fluctuations were taken according to $a = b = 0.01$ and $c_s^2 = 0.02$.

Figure 2 (lower panel) clearly shows parametric resonance in the case of spike-like oscillations at $\Omega_2 = \Omega/2 \simeq 0.5013$ which corresponds to the second mode of the resonance, see Eq. (4.9b), with the chosen values of parameters: $a = b = 0.01$ and $c = 0.02$. For these values μ is positive and equal to $1.47 \cdot 10^{-4}$. These values of a, b and c were merely chosen to produce the figures showing the resonant behavior, which however appears for a very wide range of these parameters. On the other hand, parametric resonance is more sensitive to variations of y_0 and of course of $\Omega_{1,2}$. In fact, the exponential growth is slower as we depart from the resonance frequency.

4.3 Antifriction amplification

Antifriction amplification of perturbations is induced by the change in the signs of the coefficients in front of the odd derivatives in the 4th order equation. If they become

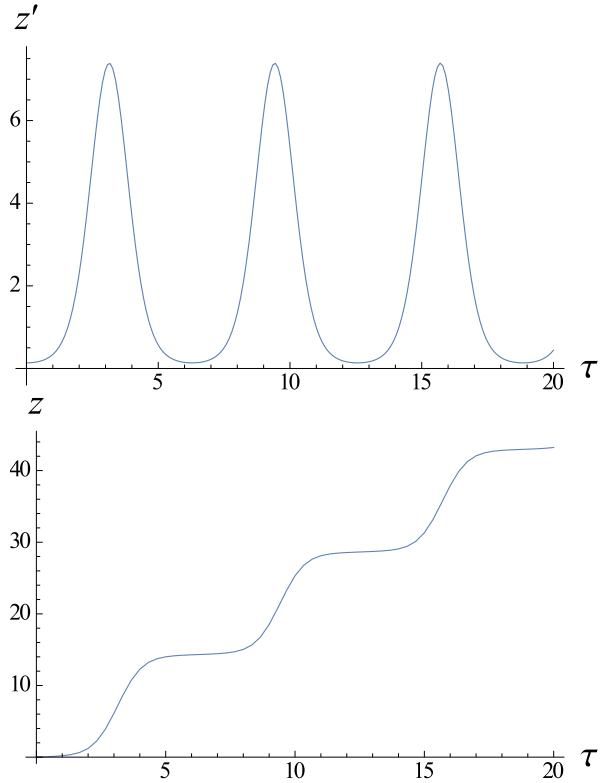


Figure 3 The function $z(\tau)$ (*lower panel*) and its derivative $z'(\tau)$ (*upper panel*), as given by Eq. (4.28) for $C_0 = 1$, $C_1 = C_2 = 0$, $y(\tau) = y_0 \cos(\tau)$, and $\alpha y_0 = 2$.

negative (and they do), then instead of damping the oscillations they would lead to their enhancement, as can be easily seen solving the oscillator equation with $\Gamma < 0$:

$$\ddot{z} + \Gamma \dot{z} + \omega^2 z = 0\,. \tag{4.25}$$

In Section 4.1 we considered the case of small amplitude of the oscillating background and saw that Eq. (4.10) can be solved analytically. It is interesting that this equation can be also solved analytically in the opposite limit of large amplitude of oscillations, but with small c and μ. In this case Eq. (4.10) becomes

$$z'''' + \alpha y' z''' + 2\alpha z'' y'' + \alpha z' y''' = z'''' + \alpha (z' y')'' = 0\,. \tag{4.26}$$

The equation is easily integrated:

$$z'' + \alpha z' y' = C_1 + C_2 \tau\,, \tag{4.27}$$

leading to the solution:

$$z' = C_0\, e^{-\alpha y(\tau)} + C_1\, e^{-\alpha y(\tau)} \int_0^\tau d\tau_1\, e^{\alpha y(\tau_1)} + C_2\, e^{-\alpha y(\tau)} \int_0^\tau d\tau_1 \tau_1 e^{\alpha y(\tau_1)}\,, \tag{4.28}$$

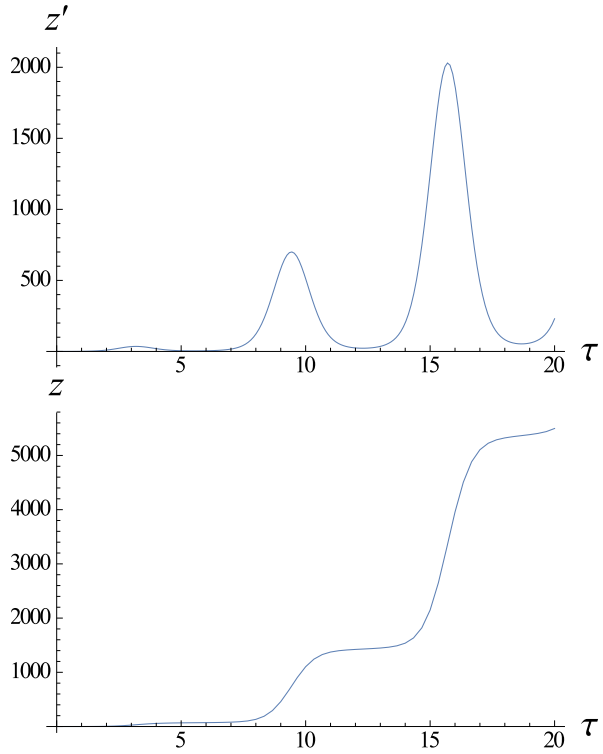


Figure 4 The function $z(\tau)$ (*lower panel*) and its derivative $z'(\tau)$ (*upper panel*), as given by Eq. (4.28) for $C_2 = 1$, $C_0 = C_1 = 0$, $y(\tau) = y_0 \cos(\tau)$, and $\alpha y_0 = 2$.

or equivalently

$$z = z_0 + \int^{\tau} d\tau_1 \left[C_0\, e^{-\alpha y(\tau_1)} + e^{\alpha y(\tau_1)} \int^{\tau_1} d\tau_2\, (C_1 + C_2\tau_2)\, e^{\alpha y(\tau_2)} \right] . \qquad (4.29)$$

We take for definiteness $y(\tau) = y_0 \cos(\Omega_1 \tau)$. From the expression (4.28) it is clear that the derivative z' is small when $\alpha y < 0$, while z' is positive and large for $\alpha y > 0$, so the function z remains constant or grows in the first and in the second case respectively.

This behaviour is shown in Figs. 3, 4. The amplification of $z(\tau)$ takes place independently of the frequency of the background curvature oscillations Ω_1. This can be explained by the change of the sign of the coefficients in front of the odd derivatives in Eqs. (4.10) and (4.26) for large y_0. When these coefficients are positive they act as a friction force but when they are negative they act as antifriction. This effect is showed numerically below for the full equation (4.10).

Here we solve Eq. (4.10) numerically for the ad-hoc chosen parameters $a = 0.01$, $b = 0.01$, $c = 0.02$, for different values of y_0 and $\Omega_{1,2}$. The antifriction amplification is observed at the frequencies away from the resonance values, if y_0 exceeds a threshold value, y_{th}. The farther away the frequency is from the resonance, the larger the threshold. For example for harmonic oscillations with $\Omega_1/\Omega = 3.2$ the threshold

value is $y_{th} = 169$, while for $\Omega_1/\Omega = 4.4$ the threshold is $y_{th} = 267$. The evolution of $z(\tau)$ in the first case is depicted in Fig. 5 (upper panel).

In the lower panel of Fig. 5 we present the evolution of $z(\tau)$ for spike-like oscillations with out-of-resonance frequencies $\Omega_2/\Omega = 0.6$. Similar to the case of harmonic $y(\tau)$, the farther away the frequency is from the resonant one, approximately equal to 0.5, the larger the threshold value of y_0 necessary for generating an unstable solution. The magnitudes of y_0 are quite close to the threshold values, $y_0 = 400$ for $\Omega_2/\Omega = 0.6$ and $y_0 = 702$ for $\Omega_2/\Omega = 0.7$. Note that the threshold value of y_0 at the resonance with $\Omega_2/\Omega \approx 0.5$ is about 15.

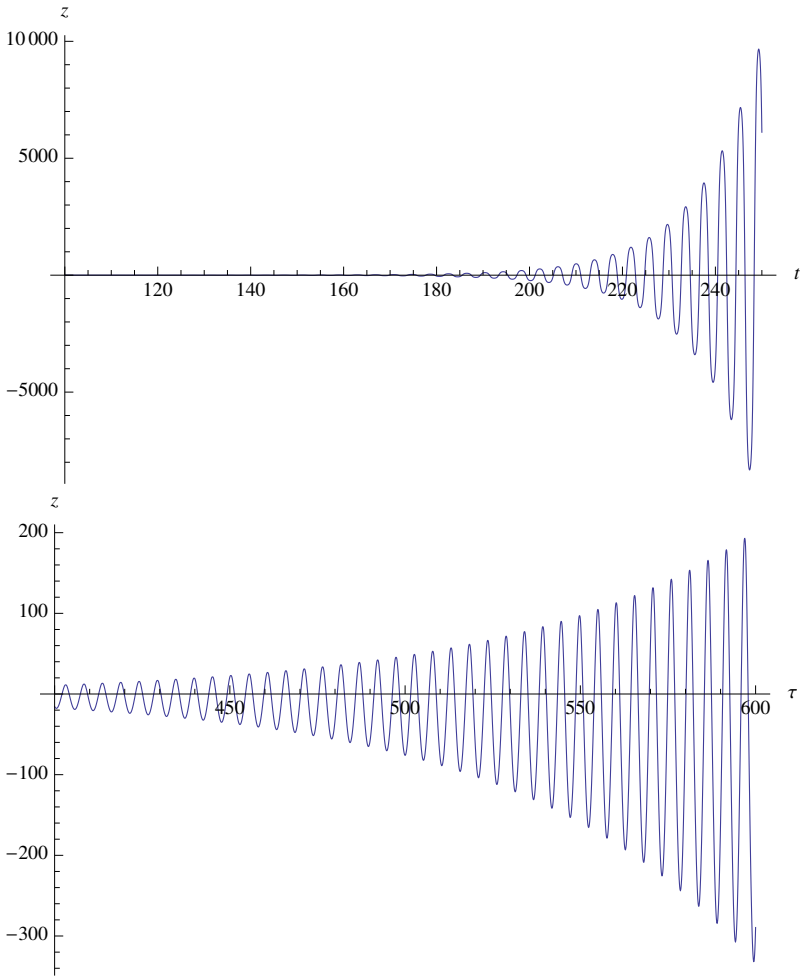


Figure 5 *Upper panel*: Antifriction effect in evolution of $z(\tau)$ for harmonic curvature oscillations with $y_0 = 169$, $\Omega_1/\Omega = 3.2$. *Lower panel*: Antifriction effect in evolution of $z(\tau)$ for spike-type curvature oscillations with $y_0 = 400$, $\Omega_2/\Omega = 0.6$.

5 Discussion and Conclusions

A general feature of $F(R)$ modified gravity are high frequency oscillations of curvature and metric in contracting matter systems such as a gas cloud of interstellar or intergalactic matter in the process of star or galaxy formation. A similar effect might take place in the process of stellar collapse, e.g. before a supernova explosion and subsequent formation of neutron star or black hole. As we have shown, the evolution of metric and density perturbations in an oscillating gravitational background possesses new and very interesting features in addition to the usual exponential Jeans instability, which itself occurs at a different scale than in General Relativity.

The oscillating background metric and curvature of the system induce an effect analogous to the standard parametric resonance, resulting in an amplification of the fluctuations. There is a quantitative difference with respect to the standard case because the usual parametric resonance is described by a second order differential equation, while in modified gravity the Eq. (4.7), which governs the evolution of fluctuations, is fourth order.

We study here a rather general equation without restrictions to any specific $F(R)$-theory. We use an approach concerning the form of the oscillating metric and curvature background, considering the two cases of arbitrary harmonic oscillations and spike-like solutions of the type found in our previous work [16]. In both cases the fourth order equations (4.7) or (4.10), which govern the evolution of the density perturbations, demonstrate a very rich pattern of different types of instabilities. There is a close analogue of parametric resonance, which is easy to describe theoretically,

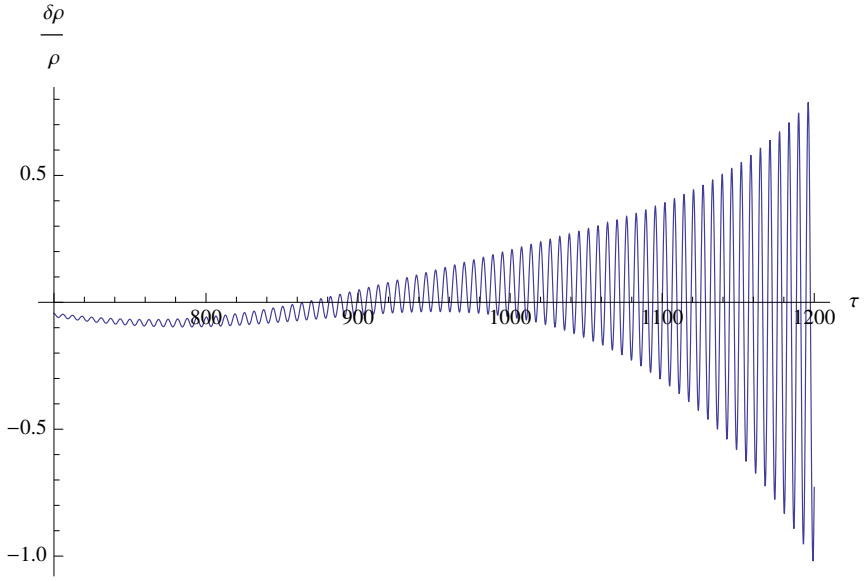


Figure 6 Evolution of $\delta\varrho/\varrho$ in parametric resonance region induced by the spike-like solution with $y_0 = 30$, $\Omega_2/\Omega = 0.5$. The initial value of metric perturbation is taken as $\delta B(0) \equiv z(0) = 10^{-3}$.

almost in the same way as the usual parametric resonance. However, in addition to this we found a new kind of instability induced by negative signs of the coefficients in front of the odd derivatives in the equation – we dub it the antifriction instability. The latter effect occurs at rather high amplitude of curvature oscillations, higher than the background GR value. Systems where curvature oscillations with large amplitude are induced have been found in our work [16]. Of course these results are reliable only if perturbations are sufficiently small, not larger than unity when written in dimensionless form.

We have studied Eq. (4.10) analytically and numerically, solving for the perturbation in the space–space metric term. Based on these solutions, one can calculate the evolution of the relative density contrast using Eq. (4.11). Barring accidental cancellations, exponentially growing solutions for z will lead to an equivalent behaviour for the density perturbation $\delta\varrho$, as shown in Fig. 6 in the case of parametric resonance induced by the spiky solution with moderately large amplitude $y_0 = 30$.

We can see that in a relatively short time, about $10^3\omega^{-1}$, the density perturbation would reach unity if the initial metric perturbations are about 10^{-3}. The effective time can be much shorter than the gravitational time typical for the Jeans-type rise of density perturbations, since ω is usually large. Though the density contrast is an oscillating function of time, its impact on structure formation may be non-negligible and should lead to constraints on the parameters of the underlying $F(R)$-theory. This will be studied elsewhere.

Another potentially interesting effect of the oscillating rise of curvature perturbations is the generation of curvature propagation around contracting bodies, i.e. a scalar mode for gravitational waves having mass equal to ω, which may vary from practically zero up to the scalaron mass m. They may lead to a considerable loss of energy of the collapsing objects.

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