Astronomical and Astrophysical Transactions (*AApTr*), 2014, Vol. 28, Issue 4, pp. 331–336,
ISSN 1055-6796, Photocopying permitted by license only, ©Cambridge Scientific Publishers

Stäckel models of galaxies with generalized isothermal potential

A.O. Gromov*

Department of Space Technologies and Applied Astrodynamics, St. Petersburg State University, 35 Universitetskij Pr., Staryj Peterhof, St. Petersburg 198504, Russia

Received 13 October, 2013

In this paper a new family of rotationally symmetric models of mass distribution in a stellar system is proposed. The potential of this family is of Stäckel type. For this potential a quadratic in velocities integral of motion exists, hence velocity distribution in such models can be triaxial. The generalized isothermal potential is considered and a self-gravitating model is constructed by Kuzmin's method.

Keywords: Stellar dynamics – galaxies: mass distribution – galaxies: Stäckel models

1 Introduction

In this paper a family of rotationally symmetric models of mass distribution in stellar systems is considered. We suppose that the third quadratic integral of motion

$$I_3 = (Rv_z - zv_R)^2 + z^2 v_\theta^2 + z_0^2(v_z^2 - 2\Phi^*)$$

exists. Here R, z are cylindrical coordinates, z_0 constant with the dimension of length, Φ^* is function related to potential Φ by

$$\begin{cases} z_0^2 \dfrac{\partial \Phi^*}{\partial R} = z^2 \dfrac{\partial \Phi}{\partial R} - Rz \dfrac{\partial \Phi}{\partial z}, \\ z_0^2 \dfrac{\partial \Phi^*}{\partial z} = (R^2 + z_0^2) \dfrac{\partial \Phi}{\partial z} - Rz \dfrac{\partial \Phi}{\partial R}. \end{cases}$$

Then the potential has the following form:

$$\Phi = \frac{\varphi(\xi_1) - \varphi(\xi_2)}{\xi_1^2 - \xi_2^2},$$

*Email: granat08@yandex.ru

where $\xi_1 \in [1;\infty)$, $\xi_2 \in [-1;1]$ are elliptic coordinates defined by formulas

$$R = z_0\sqrt{(\xi_1^2-1)\,(1-\xi_2^2)}\,, \qquad\qquad z = z_0\xi_1\xi_2\,.$$

Function φ is chosen so that density is nonnegative and the mass of the model is finite.

The theory of constructing models with the third quadratic integral of motion was developed by Kuzmin (1956a, 1956b). Such models were constructed earlier, e.g., by Kuzmin and Malasidze (1987).

In this paper we try to construct a non-spherical model with the generalized isothermal potential.

2 Potential and density of the model

In the equatorial plane the generalized isothermal potential has the following form:

$$\Phi(R,0) = \Phi_0 \ln\left(1+\frac{\beta}{w(R)}\right)\,,$$

where Φ_0, $\beta \in [0;+\infty)$ are parameters of the model, the function $w(R) > 0$ is defined as

$$w^2(R) = 1 + \varkappa^2\frac{R^2}{R_0^2}.$$

Here $\varkappa$, R_0 are parameters of the model. For simplicity we assume that $\varkappa^2 = O(\beta^2)$, $R_0 = 1$.

This potential was proposed by Kuzmin, Veltmann and Tenjes for a spherical system (1986). They found that the density of this model can approximate the de Vaucouleurs profile.

It is easy to find that

$$\varphi(\xi) = \xi^2\Phi_0 \ln X\,,$$

$$X = 1 + \frac{\beta}{\sqrt{1+\varkappa^2 z_0^2(\xi^2-1)}}\,.$$

Then we obtain the following expression for spatial density

$$4\pi G\rho = \Phi_0\left(\frac{-2\dfrac{\Phi}{\Phi_0}(2-\xi_1-\xi_2)+\mu(\xi_1)+\mu(\xi_2)}{(\xi_1^2-\xi_2^2)^2} + 2\frac{\big(\nu(\xi_1)-\nu(\xi_2)\big)(2-\xi_1-\xi_2)}{(\xi_1^2-\xi_2^2)^3}\right),$$

where

$$\mu(\xi) = (\xi^2-1)\,[A_1+A_2+A_3]\,,$$

$$A_1 = 2\ln X\,,$$

$$A_2 = -\frac{5\xi^2\beta\varkappa^2 z_0^2(1-\varkappa^2 z_0^2) - 2\xi^4\beta\varkappa^4 z_0^4}{\left(1+\varkappa^2 z_0^2(\xi^2-1)\right)^{5/2}\cdot X},$$

$$A_3 = -\frac{\xi^4\beta^2\varkappa^4 z_0^4}{\left(1+\varkappa^2 z_0^2(\xi^2-1)\right)^{3}\cdot X^2},$$

$$\nu(\xi) = \frac{\xi^4\beta\varkappa^2 z_0^2}{\left(1+\varkappa^2 z_0^2(\xi^2-1)\right)^{3/2}\cdot X}.$$

Below some graphs are presented. In all graphs $\Phi_0 = 1$ and $\varkappa = \beta$. ρ_0 is the value of the central density for a model with $z_0 = 1.5$, $\beta = 1$.

Because uncertainty $\dfrac{0}{0}$ exists, the formula by Rodionov (1974) was used for constructing graphs of density on the axis of symmetry.

Moreover the density on the axis of symmetry and in the equatorial plane decreases. It is physically plausible.

3 Density asymptotics

Consider the behavior of density at an infinite distance from the centre. Let $\xi_2 = 0$, the expression for density will be as follows:

$$4\pi G\rho = 2\frac{\left(\varphi(\xi) - \xi\varphi'(\xi)\right)\left(2-\xi^2\right)}{\xi^6} - \frac{\left(\xi^2-1\right)\varphi''(\xi)}{\xi^4}.$$

We omit the lower index at ξ_1. We notice that $\varphi(0) = 0$.

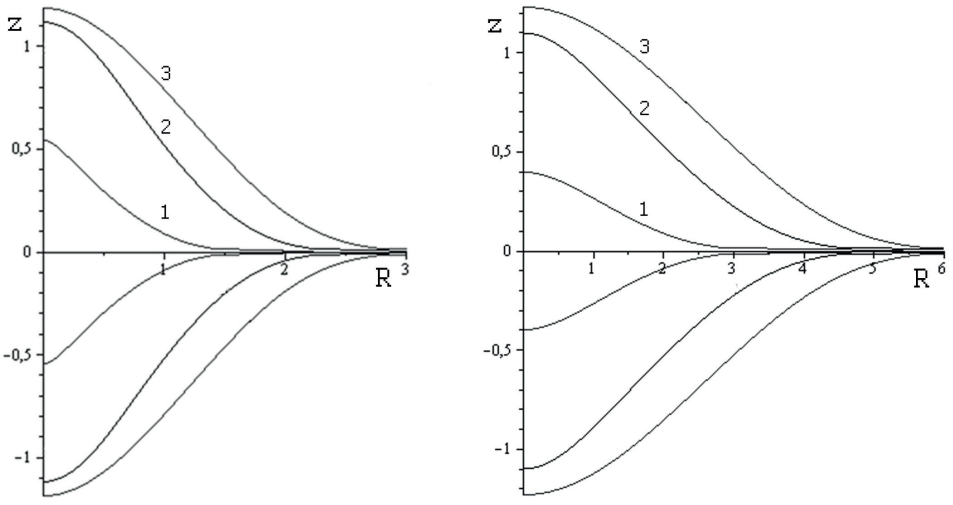


Figure 1 Equidensities: $z_0 = 1.5$, $\beta = 2$ (left) and $z_0 = 3$, $\beta = 1$ (right); 1, $\rho = 0.3\rho_0$; 2, $\rho = 0.1\rho_0$; 3, $\rho = 0.05\rho_0$.

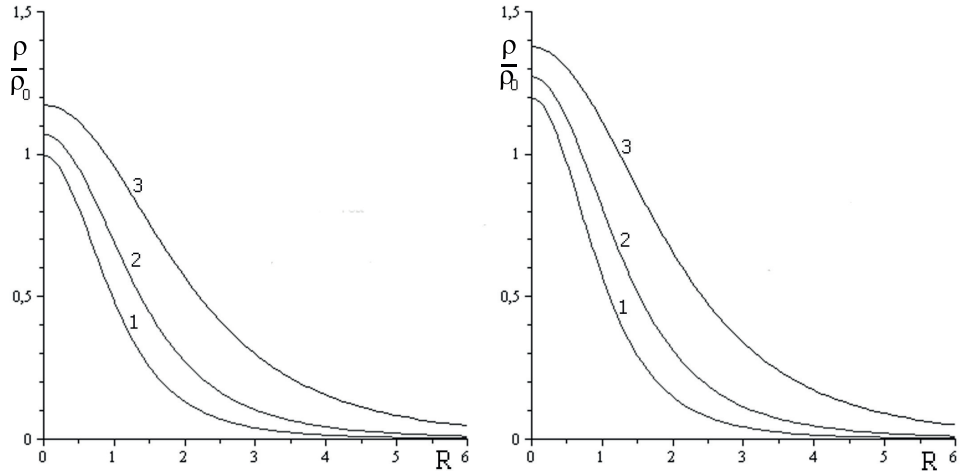


Figure 2 Density run in the equatorial plane: $\beta = 1$ (left) and $\beta = 2$ (right); 1, $z_0 = 1.5$; 2, $z_0 = 2$; 3, $z_0 = 3$.

If $\xi_2 = 0$, then

$$R^2 = z_0^2 \left(\xi^2 - 1\right) ,$$

and $\xi = O(R)$.

We substitute φ in the expression for density in the equatorial plane

$$\frac{4\pi G}{\Phi_0} \cdot \rho = \frac{2 \cdot (2 - \xi^2)}{\xi^6} \cdot [B_1 + B_2] - \frac{\left(\xi^2 - 1\right)}{\xi^4} \cdot [C_1 - C_2 - C_3] ,$$

with

$$B_1 = -\xi^2 \ln X ,$$

$$B_2 = \frac{\xi^4 \beta \varkappa^2 z_0^2}{\left(1 + \varkappa^2 z_0^2 (\xi^2 - 1)\right)^{3/2} \cdot X} ,$$

$$C_1 = 2 \ln \cdot X ,$$

$$C_2 = \frac{2\xi^4 \beta \varkappa^4 z_0^4}{\left(1 + \varkappa^2 z_0^2 (\xi^2 - 1)\right)^{5/2} \cdot X} ,$$

$$C_3 = \frac{\xi^4 \beta^2 \varkappa^4 z_0^4}{\left(1 + \varkappa^2 z_0^2 (\xi^2 - 1)\right)^3 \cdot X^2} .$$

Using $\xi \to \infty$ and expansion in the Taylor's series $\ln\left(1 + \frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + +o\left(\frac{1}{x^2}\right)$, we find that

$$\frac{4\pi G}{\Phi_0} \cdot \rho \sim \frac{2}{\xi^6} \cdot \left[-\xi \cdot \frac{\beta}{\varkappa z_0} + \frac{\beta^2}{2\varkappa^2 z_0^2} + \xi^2 \cdot o\left(\frac{1}{\xi^2}\right) + \frac{\xi^4}{\xi^3} \cdot \frac{\beta}{\varkappa z_0} \right] \left(2 - \xi^2\right) -$$

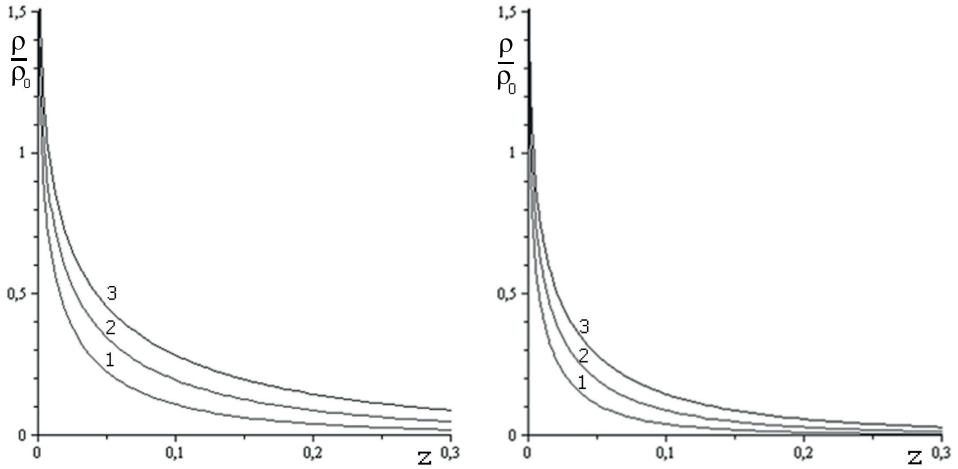


Figure 3 Density run on the axis of symmetry: $z_0 = 1.5$ (left) and $z_0 = 3$ (right); 1, $\beta = 1$; 2, $\beta = 2$; 3, $\beta = 3$.

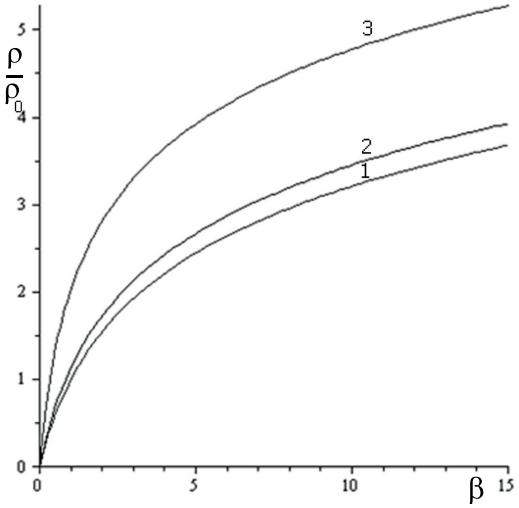


Figure 4 Dependence of density in the centre of model on β: 1, $z_0 = 1.5$; 2, $z_0 = 2$; 3, $z_0 = 3$.

$$-\frac{1}{\xi^4}\cdot\left[\frac{2}{\xi}\cdot\frac{\beta}{\varkappa z_0}-\frac{\beta^2}{\varkappa^2 z_0^2\xi^2}+o\left(\frac{1}{\xi^2}\right)-\frac{2\xi^4}{\xi^5}\cdot\frac{\beta}{\varkappa z_0}-\frac{1}{\xi^2}\cdot\frac{\beta^2}{\varkappa^2 z_0^2}\right]\left(\xi^2-1\right).$$

It is equivalent to $\dfrac{1}{\xi^4}\cdot\dfrac{\beta^2}{\varkappa^2 z_0^2}+\dfrac{1}{\xi^2}\cdot o\left(\dfrac{1}{\xi^2}\right)$. Hence, density decreases as ξ^{-4} and as $\rho = O\left(R^{-4}\right)$. Such an order of decreasing provides the mass of the model to be finite.

4 Summary

In this paper, models of mass distribution in a stellar system with generalized isothermal potential are proposed.

Because of the existing third integral of motion, the velocity ellipsoid of such a model can be triaxial, which agrees with observations. For our model, the potential is of Stäckel type, so the Hamilton–Jacobi equation is separable. The task of finding the orbits is reduced to quadratures.

Changing the parameters of the model, we can construct stellar systems of different oblateness.

The expression for spatial density is analytical. Through this we can explore some tasks, e.g., to study the stability of this model.

References

Gromov A.O. Control Processes and Stability XLIII Conference of Faculty Appl. Math. Control Proc. Saint-Petersburg Univ. Press, Saint-Petersburg, 125 (2012a).
Gromov A.O. Astron. Tsirk., **1579** (2012b).
Kuzmin G.G. Publ. Tartu astron. obs., **32** 332 (1956a).
Kuzmin G.G. Astr. Zh. **33** 27 (1956b).
Kuzmin G.G., Malasidze G.A. Publ. Tartu astron. obs. **52** 48 (1987).
Kuzmin G.G., Veltmann Ü.-I.K., Tenjes P.L. Publ. Tartu astron. obs. **51** 232 (1986).
Rodionov V.I. Vestn. Leningrad. Univ. **13** 142 (1974).