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Cluster dynamics in the galactic tidal field

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The dynamics of star cluster moving along a circular orbit in the axisymmetric steady Galaxy is considered. The hierarchy of gross-dynamic equations is written down, and some theorems on cluster dynamics are proved.

Keywords: Stellar dynamics – methods: analytical – clusters: evolution

1 Introduction

Dynamical evolution of real star clusters is governed by complicated interfacing of many factors, both internal and external (e.g. Kholopov 1981; Spitzer 1987). Following the classical paper by Bok (1934) we will analyze only two of these factors, namely, self-gravitation and galactic tidal force. Bok (1934) considered a cluster moving along a circular orbit in the steady axisymmetric galactic field. Earlier a similar model was investigated by Roche (1859), Picart (1904) and many others who studied dynamics of the protoplanetary cloud or atmospheres of comets. These works were critically reviewed by Rein (1936a). Fessenkov (1951) discussed the role of tidal effects in cosmogony and wrote about possible applications of his theory to cluster dynamics.

Bok (1934) worked with a model of star cluster as a homogeneous triaxial ellipsoid, used the tidal approximation for the galactic potential and concluded that star orbits are finite if the cluster density

$$\varrho > \varkappa_\varpi^2 / (\pi G \beta), \tag{1}$$

where G is the gravitational constant, $\varkappa_\varpi^2 = 4A(A-B)$, A, B being Oort's dynamical coefficients, and β, a dimensionless factor depending on the shape of cluster and equal to 4/3 for spherical systems and 0.2 for discs (e.g. Chandrasekhar 1942; Ogorodnikov 1965).

Bok (1934), Rein (1936b) and Cimino (1956) discussed the Hill stability for stars of a cluster. The critical density of a cluster can be estimated from the condition that the cluster of a given mass lies inside the Hill critical surface. It can be easily

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found (Nezhinsky & Ossipkov 1987; Ossipkov 2006) that the critical density must be larger almost by order of magnitude than according to condition (1), in contrast to the original conclusion by Bok (1934). But it can be shown (Ossipkov 2007; see below) that Bok's condition (1) is valid as a condition for the centre of cluster to be stable as a libration point, ϱ being now the central density. Ogorodnikov (1965) used the Bok model for studying dissolution of sparse star clusters in the galactic field.

Numerical orbit calculations in the field of a cluster and the Galaxy were fulfilled by Jefferys (1976), Carpintero et al. (1999) and Davydenko (2013), who found that retrograde orbits are more stable, and formation of tails of clusters may be due to tidal effects. Earlier such tails were studied in the course of N-body experiments (e.g. Chumak & Rastorguev 2005).

But it must be noted that the problem studied in these works was not self-consistent. The gravitational potential of cluster was supposed to be constant though stars could escape from the cluster. The first self-consistent analysis was fulfilled by Mineur (1939). He considered the homogeneous triaxial gravitating ellipsoid with isotropic velocity distribution and sought conditions for such system to be in equilibrium in the galactic field. Mineur (1939) found that for such model the condition of its existence has the form (1) with a factor β depending on $\varkappa_\varpi^2/\varkappa_z^2$, where $\varkappa_z$ is the frequency of small vertical star oscillations in the galactic field (the Kuzmin parameter). $\beta \approx 4$ for the Solar neighbourhood of our Galaxy. Mayot (1945) applied the classical theory of equilibrium figures to Mineur ellipsoids. Nezhinsky (1985) proved that the stable system of 'drops' of ideal fluid can exist in galaxies (see also Ossipkov et al. 1997). Kondratiev (2001) developed a similar theory to find the equilibrium shape of globuls. The theory was generalized by Kondratiev and Trubitsina (2010) for ellipsoids with internal flows.

Van Wijk (1949) and Kuzmin (1963) generalized the theory of Mineur (1939) for non-homogeneous clusters with isotropic velocity distribution.

A more general analysis of equlibrium of cluster (with anisotropic velocity distribution) was done by Ossipkov (2006). His analysis was based on equations of gross-dynamics (Ossipkov 1993, 2001). In this paper we study the non-stationary gross-dynamics of cluster.

2 The Bok problem

At first we recall equations of star motion for a cluster moving with an angular velocity Ω at the distance ϖ_0 from the galactic centre. Let $\mathbf{x} = (x, y, z)$ be a rotating frame of reference, an axis x being directed to the galactic anticentre. Then

$$\ddot{\mathbf{x}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{x}} + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times (\mathbf{R}_0 + \mathbf{x}) = \nabla(\Phi_c + \Phi_g). \tag{2}$$

Here $\mathbf{R}_0 = (\varpi_0, 0, 0)$, $\Phi_c(\mathbf{x}, t)$ is a (positive) potential of a cluster, $\Phi_g(\varpi, z)$ a (positive) galactic potential. We set

$$\Phi_g(\varpi, z) = \Phi_0 + k_1(\varpi - \varpi_0) + k_2(\varpi - \varpi_0)^2 + k_3 z^2$$

(the tidal approximation). It is evident that $k_1 = -\Omega^2\varpi_0$. Denote

$$\varkappa_\varpi^2 = \Omega^2 + 2k_2 = 4A(A - B), \qquad \varkappa_z^2 = -2k_3.$$

Then Eqs (2) can be written in the following form:

$$\ddot{x} - 2\Omega\dot{y} = \partial\Phi_c/\partial x + \varkappa_\varpi^2 x\,, \tag{3}$$

$$\ddot{y} + 2\Omega\dot{x} = \partial\Phi_c/\partial y\,, \tag{4}$$

$$\ddot{z} = \partial\Phi_c/\partial z - \varkappa_z^2 z \tag{5}$$

(e.g. Ogorodnikov 1965; Ossipkov 2007; Buliga & Ossipkov 2011; Proskurin & Ossipkov 2013). Now define dimensionless coordinates $\xi = x/r_0$, $\eta = y/r_0$, $\zeta = z/r_0$ with $r_0 = (GM_c/\varkappa_\varpi^2)^{1/3}$ (M_c is the mass of cluster) and dimensionless time $\tau = t\varkappa_\varpi$ and write down Eqs (3)–(5) in a dimensionless form:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\xi - \gamma\frac{\mathrm{d}}{\mathrm{d}\tau}\,\eta = \frac{\partial\varphi}{\partial\xi} + \xi\,, \tag{6}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\eta + \gamma\frac{\mathrm{d}}{\mathrm{d}\tau}\,\xi = \frac{\partial\varphi}{\partial\eta}\,, \tag{7}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\zeta = \frac{\partial\varphi}{\partial\zeta} - k\zeta\,. \tag{8}$$

Here $\varphi(\xi,\eta,\zeta) = \Phi_c(\xi r_0, \eta r_0, \zeta r_0)/(GM_c r_0^{-1})$ is a dimensionless potential of cluster, $\gamma = 2\Omega\varkappa_R^{-1}$ and $k = (\varkappa_z/\varkappa_R)^2$ dimensionless factors. Note that $\Omega \approx 27$ km$\cdot$s$^{-1}\cdot$ kpc^{-1}, $\varkappa_\varpi \approx 42$ km$\cdot$s$^{-1}\cdot$kpc^{-1}, $\varkappa_z \approx 85$ km$\cdot$s$^{-1}\cdot$kpc^{-1} for the Solar neighbourhood. Then $\gamma \approx 1.25$, $k \approx 4$. The author called the factor $\varkappa_\varpi$ the *tidal increment* and the problem of solving these equations the *Bok problem* (Ossipkov 2003).

If the cluster potential is steady, the Jacobi integral exists for Eqs (6)–(8),

$$H = \frac{1}{2}\left[\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}\right)^2\right] - \varphi - \frac{1}{2}(\xi^2 - k\zeta^2). \tag{9}$$

If a cluster is a homogeneous ellipsoid, Eqs (6)–(8) are linear ones, and two other integrals of motion can be written down (Mineur 1939). But it seems very probable that no more global integrals of motion exist in general, and orbits can be chaotic, as calculations by Jefferys (1970) and Carpintero et al. (1999) show. The search for conditions of orbit chaotization is the *internal Bok problem*. As the first step to its analysis the author tried to investigate libration points for Eqs (6)–(8) (Ossipkov 2007).

It was found that all libration points lie on the ξ-axis. If a cluster is spherical, $\varphi(\xi,\eta,\zeta) = f(r)$, $r = (\xi^2+\eta^2+\zeta^2)^{1/2}$, they can be found from the equation

$$f'(|\xi|) + |\xi| = 0.$$

At least one pair of libration points exists for plausible models of cluster. A condition of their stability was found. Symmetric libration points were found to be unstable for potentials of Schuster–Plummer and Idlis.

It is evident that the centre of cluster $\xi = \eta = \zeta = 0$ is a libration point. An analysis of its stability (solving linearized equations of motion) is reduced to Bok's solution of Eqs (6)–(8) for a homogeneous cluster. Then the necessary condition of its stability has the form (1), ϱ being a central density of cluster. Hence, one can

expect from the KAM-theory that stellar orbits nearby the centre are ordered. As we can judge, transition to chaos for the internal Bok problem was not studied yet.

Let $\xi = a > 0$, $\eta = 0$, $\zeta = 0$ are coordinates of the libration point. The corresponding value of the Jacobi integral (9)

$$H_a = -\varphi_a - a^2/2, \qquad \varphi_a = \varphi(a, 0, 0).$$

The zero-velocity surface

$$S_a = \big\{(\xi, \eta, \zeta) \,\big|F(\xi, \eta, \zeta) = H_a\big\},$$

where $F(\xi, \eta, \zeta) = -\varphi - (\xi^2 - k\zeta^2)/2$ (the *Hill critical surface*). Points of intersection of S_a with η-axis, η_a, and with ζ-axis, ζ_a, are solutions of equations

$$\varphi(0, \eta_a, 0) = \varphi_a + a^2/2, \qquad \varphi(0, 0, \zeta_a) = k\zeta^2 + \varphi_a + a^2/2.$$

The following assertions are valid:

If $(0, \eta_a, 0) \in S_a$ then $(0, -\eta_a, 0) \in S_a$.

If $(0, 0, \zeta_a) \in S_a$ then $(0, 0, -\zeta_a) \in S_a$.

If the Jacobi constant $H < H_a$ then the stars will be retained inside the Hill surface. What will be with stars when $H > H_a$? This is the *external Bok problem*. Approximate conditions for stars to be bounded with a cluster were suggested by Ross et al. (1997) and Davydenko (2008). Extensive calculations of orbits for this problem were fulfilled by Davydenko (2013) for various values of the scale parameter of the Schuster–Plummer potential.

Note, at last, that it is an unsolved and very difficult mathematical problem to find the Hill surface for clusters of infinite or large enough size.

3 Moment equations

In principle, the exhaustive description of cluster dynamics must be deduced from the kinetic equation for a distribution function $f(\mathbf{x}, \mathbf{v}, t)$,

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}}\,\mathbf{v} + \frac{\partial f}{\partial \mathbf{v}}\,\dot{\mathbf{v}} = \mathrm{St} f.$$

Substituting Eqs (3)–(5) yields

$$\begin{aligned}\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \left(\frac{\partial \Phi_c}{\partial x} + \varkappa_\varpi^2 x + 2\Omega v\right)\frac{\partial f}{\partial u} \\ + \left(\frac{\partial \Phi_c}{\partial y} - 2\Omega u\right)\frac{\partial f}{\partial v} + \left(\frac{\partial \Phi_c}{\partial z} - \varkappa_z^2 z\right)\frac{\partial f}{\partial w} = \mathrm{St} f.\end{aligned} \tag{10}$$

According to the Jeans theorem, a steady distribution function depends only on time-independent isolating integrals of Eqs (3)–(5). Such integrals are known only for a model of a homogeneous ellisoid, and the corresponding distribution function was found by Fellhauer & Heggie (2005) by analogy with classical ellipsoid of Freeman (1966). We guess that models by Bisnovatyi-Kogan (1986) can be used for finding

more general distributions. The approximate distribution function with ellipsoidal velocity distribution was suggested by Danilov (2006, 2008).

In general, we do not know integrals of Eqs (3)–(5). Hence, we cannot use the kinetic equation (10) for analytical studying cluster dynamics. We will try to find equations for moments of the distribution function. Neglecting star evaporation and dynamical friction of stars it is easy to find the continuity equation

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}\varrho V_x + \frac{\partial}{\partial y}\varrho V_y + \frac{\partial}{\partial x}\varrho V_z = 0 \tag{11}$$

and the equations for the streaming velocity (the generalized Jeans equations)

$$\frac{\partial}{\partial t}\varrho V_x + \frac{\partial}{\partial x}\varrho S_{xx} + \frac{\partial}{\partial y}\varrho S_{xy} + \frac{\partial}{\partial z}\varrho S_{xz} - \varrho\left(\frac{\partial \Phi_c}{\partial x} + \varkappa_R^2 x\right) - 2\Omega\varrho V_y = 0, \tag{12}$$

$$\frac{\partial}{\partial t}\varrho V_y + \frac{\partial}{\partial x}\varrho S_{xy} + \frac{\partial}{\partial y}\varrho S_{yy} + \frac{\partial}{\partial z}\varrho S_{yz} - \varrho\frac{\partial \Phi_c}{\partial y} + 2\Omega\varrho V_x = 0, \tag{13}$$

$$\frac{\partial}{\partial t}\varrho V_z + \frac{\partial}{\partial x}\varrho S_{xz} + \frac{\partial}{\partial y}\varrho S_{yz} + \frac{\partial}{\partial z}\varrho S_{zz} - \varrho\left(\frac{\partial \Phi_c}{\partial z} - \varkappa_z^2 z\right) = 0. \tag{14}$$

Here $\varrho = \int f\,\mathrm{d}^3\mathbf{v}$ is a density, $\mathbf{V} = (V_x, V_y, V_z) = \int \mathbf{v} f\,\mathrm{d}^3\mathbf{v}/\varrho$, a streaming velocity (a centroid velocity), $S_{ij} = \int \mathbf{v}_i v_j f\,\mathrm{d}^3\mathbf{v}/\varrho$, a tensor of full (!) moments of the second order of velocity distribution. Terms due to stellar encounters (omitted here) were written down by Ossipkov (1985).

Multiplying Eq. (10) by $v_i v_j$ and integrating over velocities yields the following equations (equations of micromotion, according to the terminology by Kuzmin (1965)):

$$\frac{\partial}{\partial t}\varrho S_{xx} + \cdots - 2V_x\varrho\left(\frac{\partial \Phi_c}{\partial x} + \varkappa_R^2 x\right) - 4\Omega\varrho S_{xy} = \varrho c_x,$$

$$\frac{\partial}{\partial t}\varrho S_{yy} + \cdots - 2V_y\varrho\frac{\partial \Phi_c}{\partial y} + 4\Omega\varrho S_{xy} = \varrho c_y,$$

$$\frac{\partial}{\partial t}\varrho S_{zz} + \cdots - 2V_z\varrho\left(\frac{\partial \Phi_c}{\partial z} - \varkappa_z^2 z\right) = \varrho c_z.$$

$$\frac{\partial}{\partial t}\varrho S_{xy} + \cdots - \varrho V_y\left(\frac{\partial \Phi_c}{\partial x} + \varkappa_R^2 x\right) - \varrho V_x\frac{\partial \Phi_c}{\partial y} - 2\Omega\varrho S_{yy} - 2\Omega\varrho S_{xy} = 0,$$

$$\frac{\partial}{\partial t}\varrho S_{xz} + \cdots - \varrho V_z\left(\frac{\partial \Phi_c}{\partial x} + \varkappa_R^2 x\right) - \varrho V_x\left(\frac{\partial \Phi_c}{\partial z} - \varkappa_z^2 z\right) = 0,$$

$$\frac{\partial}{\partial t}\varrho S_{yz} + \cdots - \varrho V_z\frac{\partial \Phi_c}{\partial y} - \varrho V_y\left(\frac{\partial \Phi_c}{\partial z} - \varkappa_z^2 z\right) = 0.$$

Here "$\cdots$" means terms with derivatives of third order moments, c_i are due to changes of the shape of velocity ellipsoid resulting from stellar encounters. If an

energy of macroscopic volume element conserves in spite of action of irregular forces, then $c_x + c_y + c_z = 0$.

Adding some closing relations it is possible to try to solve these equations and to construct a hydrodynamic model of star cluster in the galactic field. For instance, Mineur's (1939) ellipsoids can be found as solutions of these equations for steady homogeneous systems without centroid motions and with isotropic velocity distribution. But presently this task seems to be too complicated for more realistic models. We will average moment equations over the system that yields the gross-dynamic equations. Some of these equations were written down by the author (Ossipkov 2001). Gross-dynamics of star clusters was studied also by Danilov (1982, 2008), Danilov & Putkov (2012) and Kojanov (2003).

Multiplying the continuity equation Eq. (11) by $x_i x_j$ and averaging over the cluster yields the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{ij} = L_{ij} + L_{ji}, \tag{15}$$

where $I_{ij} = M_c\langle x_i x_j\rangle$, $L_{ij} = M_c\langle x_i V_j\rangle$, and brackets mean averaging over the system.

The following equations (the generalized tensor equations of Lagrange–Jacobi) are consequences of Eqs (12)–(14):

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{ix} = K_{ix} + W_{ix} + \varkappa_R^2 I_{ix} + 2\Omega L_{iy}, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{iy} = K_{iy} + W_{iy} - 2\Omega L_{ix}, \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{iz} = K_{iz} + W_{iz} - \varkappa_z^2 I_{iz}. \tag{18}$$

Here $K_{ij} = M_c\langle S_{ij}\rangle$ is the tensor of (total) kinetic energy, $W_{ij} = M_c\langle x_i \partial\Phi/\partial x_j\rangle$ is the tensor of potential energy.

Averaging the equations of micromotion leads to equations for the tensor of kinetic energy:

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{xx} = 2H_{xx} + 2\varkappa_R^2 L_{xx} + 4\Omega K_{xy} + M_c < c_x >, \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{yy} = 2H_{yy} - 4\Omega K_{xy} + M_c < c_y >, \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{zz} = 2H_{zz} - 2\varkappa_z^2 L_{zz} + M_c < c_z >, \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{xy} = H_{xy} + H_{yx} + 2\varkappa_R^2 L_{xy} + 2\Omega(K_{yy} - K_{xx}), \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{xz} = H_{xz} + H_{zx} + \varkappa_R^2 L_{xz} - \varkappa_z^2 L_{zx} + 2\Omega K_{yz}, \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{yz} = H_{yz} + H_{zy} - \varkappa_z^2 L_{zy} - 2\Omega K_{xy}, \tag{24}$$

where $H_{ij} = M_c\langle V_i \partial\Phi/\partial x_j\rangle$ is the 'H-tensor' by Som Sunder & Kochhar (1986). The sum of equations (19), (20), (21) yields the Jacobi integral for a cluster (generalizing the energy conservation) in the same form as in Chandrasekhar (1942), though Chandrasekhar dealt with the N-body problem.

4 Some theorems

Some general conclusions can be drawn from gross-dynamic equations.

Theorem 1. *If $I_{ij} = 0, i \neq j$, then $L_{ij} = -L_{ji}$.*

Proof follows immediately from Eqs (15).

Theorem 2. *If $(I_{xx} - I_{yy})^{\cdot} = 0$* (e.g. in the case of rotational symmetry), *then $L_{xx} = L_{yy}$.*

Proof. Subtract the equation for I_{yy} from the equation for I_{xx}.

Theorem 3. *If a system rotates around the z-axis as a rigid body, then $I_{xy} =$* const.

Proof follows from relations $L_{xy} = -\omega I_{xy}$, $L_{yx} = \omega I_{xy}$.

Theorem 4. *If $I_{xy} = 0$ then*

$$H_z = L_{xy} - L_{yx} + 2\Omega(I_{xx} + I_{yy}) = \text{const}.$$

Proof. It follows from Eqs (16) and (17) that

$$\dot{L_{xy}} - \dot{L_{yx}} = -2\Omega(L_{xx} - L_{yy}) - \varkappa_\varpi^2 I_{xy}.$$

Then the theorem follows from Eqs (15).

Note 1. The integral H_z was found by Chandrasekhar (1942) for N-body system.

Note 2. It is evident that the integral H_z is a generalization of the angular momentum integral for a non-isolated system.

Theorem 5. *If $I_{yz} = 0$, $L_{xz} = 0$ then $H_x = L_{yz} - L_{zy} =$* const.

Proof is the same as for Theorem 4.

We conclude also that if $H_x =$ const then, in general, $I_{yz} = 0$, $L_{xz} = 0$, for $\varkappa_z$, Ω are given external parameters of the problem.

Theorem 6. *If $I_{xz} = 0$, $L_{zy} = 0$, then $H_y = L_{xz} - L_{zx} =$* const.

Proof is the same as for Theorem 4.

Note. Theorems 5, 6 were also proved by Chandrasekhar (1942) for N-body systems. H_x, H_y, H_z can be called *Chandrasekhar integrals.*

We can conclude again that if $H_y =$ const then, in general, $I_{xz} = 0$, $L_{zy} = 0$.

Theorem 7. *If conditions of theorems 5 and 6 are fulfilled, then $K_{yz} + W_{yz} = 0$ and $K_{xz} + W_{xz} = 0$.*

Proof. It follows from Eqs (15) that $L_{xz} + L_{zx} = 0$, $L_{yz} + L_{zy} = 0$. Then $L_{xz} = L_{zx} = L_{yz} = L_{zy} = 0$ (see Theorem 1). Now we can use Eqs (16) and (19).

Corollary 1. *In this case $H_x = H_y = 0$, that is these Chandrasekhar integrals will be invariant relations.*

Corollary 2. *If $W_{xz} = W_{yz} = 0$, then the total ellipsoid for stars of cluster will hot be inclined to the galactic plane.*

Theorem 8. *If $I_{xy} = 0$, then $\dot{L_{xy}} = 2\Omega(L_{xx} - L_{yy})$. If $W_{xy} = 0$, then $K_{xy} = 2\Omega(I_{xx} - I_{yy})\dot{}$.*

Proof. It follows from Eqs (15) that $L_{xy} + L_{yx} = 0$. The theorem 3 is valid, hence $L_{xy} = H_z + L_{yx} - 2\Omega(I_{xx} + I_{yy})$. Then $L_{xy} = (1/2)H_z - \Omega(I_{xx} + I_{yy})$. Differentiating this relation yields that $\dot{L_{xy}} = 2\Omega(L_{xx} - L_{yy})$.

If both $W_{xy} = 0$ and $I_{xy} = 0$, then it follows from (16) and (17) that $K_{xy} = \dot{L_{yx}} - 2\Omega L_{yy} = \dot{L_{xy}} + 2\Omega L_{xx}$, and then $K_{xy} = -\Omega(L_{yy} - L_{xx})$. Applying Eqs (15) proves the theorem.

Corollary. *If $I_{xx} - I_{yy} = \text{const}$, then $K_{xy} = 0$.*

Lemma 1. *If $K_{xy} = 0$, $W_{xy} = 0$, then $L_{xy} = \Omega I_{xx} = C_1 = \text{const}$.*

Lemma 2. *If $K_{xy} = 0$, $W_{xy} = 0$, $I_{xy} = 0$, then $L_{xy} - \Omega I_{yy} = C_2 = \text{const}$.*

Theorem 9. *If $I_{xy} = 0$, $W_{xy} = 0$, $K_{xy} = 0$, then $I_{xx} - I_{yy} = \text{const}$.*

Proof. Taking into account lemmas we see that

$$0 = \dot{I_{xy}} = L_{xy} + L_{yx} = (C_1 + C_2) - \Omega(I_{xx} - I_{yy}).$$

It is difficult to image a cluster with $I_{xx}(t) - I_{yy}(t) = \text{const} \neq 0$. Then one can conclude about rotational symmetry of non-stationary clusters.

Combining Theorem 8 and Corollary of Theorem 7 leads to the following conclusion:

Theorem 10. *If $I_{xy} = 0$, $W_{xy} = 0$, then $I_{xx} - I_{yy} = \text{const}$ iff $K_{xy} = 0$.*

5 Concluding notes

We discussed the dynamics of clusters in the galactic field under minimal assumptions. Some additional suggestions are necessary for further development of the theory. One of them is the condition of quasi-homologoucity (Chandrasekhar & Elbert 1972):

$$(-W_{ii}) = -\frac{1}{2}\, s^2 G M_c^{5/2} I_{ii} / (I_{xx} + I_{yy} + I_{zz})^{3/2}, \qquad s = \text{const},$$

$$H_{ii} = -\frac{1}{4}\, s^2 G M_c^{5/2} \dot{I_{ii}} / (I_{xx} + I_{yy} + I_{zz})^{3/2}.$$

Then the system of gross-dynamic equations will be closed. The cluster exerts small oscillations relative to the equilibrium (Ossipkov 1993, 2001; Danilov 2008; Danilov & Putkov 2012). Danilov (1983, 1984) suggested to consider spherical or ellipsoidal clusters as a set of homogeneous and homological subsystems. As an example Danilov & Putkov (2012) studied oscillations in a cluster consisting of a core and a corona.

A further generalization of the theory must include analysis of dynamics of clusters on non-circular orbits (Rastorguev & Surdin 1978; Danilov 1989; Kojanov 2003). Davydenko & Ossipkov (2004) considered clusters crossing the galactic centre in the model of the Galaxy as a homogeneous spheroid. They found that an equilibrium does exist for this model and it is stable. A generalization of this study for ellipsoidal orbits can be done.

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