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Cosmic antigravitation and stability of circular orbits

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We calculate circular and epicyclic frequencies for some spherical models embedded into dark energy background and determine the zero-gravity distance and the distance from the centre where epicyclic frequency vanishes. The latter means that circular orbits become unstable at greater distances.

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1 Introduction

According to the widespread interpretation of many observational data, there exists 'dark energy' accounting for about 70% of the total energy density in the present universe, and its properties are identical to those of the cosmological term in Einstein's equations. Studies of internal kinematics for some clusters of galaxies including Virgo and the Local Group have led the researchers to conclude that outer parts of these clusters are expanding (the Hubble flow) (Karachentsev et al. 2007, 2009, 2011; Teerikorpi et al. 2008; Karachentsev & Nasonova 2010). Chernin (2001, 2008) (see also Chernin et al. (2010)) studied the possible effect of the cosmological constant on the properties of clusters of galaxies. He concluded that the presence of dark energy is responsible for the formation of the Hubble flow.

According to Chernin's model, clusters of galaxies should be considered as embedded into a homogeneous background of negative density. Hence a force acting on a galaxy consists of two parts: the attraction of other galaxies and dark matter and the repulsion due to the background. Merafina et al. (2012) investigated the effect of the negative-density background on the equilibrium of polytropic spheres and their dynamical stability. Chernin et al. (2012) generalized the virial theorem to dark energy systems (in Newtonian approximation). Chernin et al. (2010) and Bisnovatyi-Kogan & Chernin (2012) considered a cluster as a gravitationally bound spherical configuration of cold collisionless dark matter influenced also by dark energy with the cosmological density ϱ_Λ contained inside the same volume. For some models including the point-mass approximation the above authors found the zero-gravity

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radius that may be viewed as the cluster's cut-off radius. However, antigravitation due to dark energy should also act inside the cluster. In this paper we try to estimate the radius, r_*, inside which the influence of the background becomes insignificant. Also, we seek the zero-gravity radius, r_c, for some models of mass distribution. To this end, we consider circular orbits in the combined field of a gravitating sphere and antigravitating background and find conditions for their stability.

2 Circular orbits

We consider the cluster as a steady spherical system. Let $\Phi_N(r)$ be its (positive) gravitational potential. Circular orbits are known to be possible in gravitating systems at any clustercentric distance r. Contopoulos (1954) and Kurth (1955) proved that circular orbits in self-gravitating systems are stable. The potential of the background (the 'Einstein antigravity law') is equal to

$$\Phi_E(r) = \frac{4\pi}{3} G\varrho_\Lambda r^2,$$

where $-2\varrho_\Lambda$ is the effective gravitating density of dark energy (Chernin 2001; Chernin et al. 2010; Bisnovatyi-Kogan & Chernin 2012). It is evident that circular orbits in the field of the antigravitating background are unstable. The total potential is

$$\Phi_{\rm tot} = \Phi_N(r) + \Phi_E(r).$$

The circular frequency at distance r

$$\omega = \left(\frac{-1}{r} \frac{{\rm d}\Phi_{\rm tot}}{{\rm d}r} \right)^{1/2}.$$

We see that if Newtonian gravitation vanishes, circular orbits did not exist. To find the maximal possible radius of circular orbits it is necessary to solve the equation $\omega^2(r_c) = 0$, or

$$\frac{1}{r_c} \frac{{\rm d}\Phi_N}{{\rm d}r}(r_c) = \frac{8}{3}\pi G\varrho_\Lambda.$$

It is evident that r_c is the zero-gravity radius.

If Newtonian gravitation dominates at small r, circular orbits near the centre should be stable. 'Einsteinian antigravity' dominates at the periphery, hence circular orbits (when they exist) are unstable there and actual orbits must be elongated. One can suggest the existence of a critical radius $r_* < r_c$ such that all circular orbits at $r < r_*$ are stable and those at $r \in (r_*, r_c)$ are unstable. We will find r_* for some simple models.

The epicyclic frequency at clustercentric distance r is

$$\omega_r(r) = 2\omega \left(1 + \frac{r}{2\omega} \frac{{\rm d}\omega}{{\rm d}r} \right)^{1/2},$$

(see Ogorodnikov 1965; Binney & Tremaine 1987; Buliga & Ossipkov 2011). To find r_* we solve the equation $\omega_r(r_*) = 0$.

3 Models

3.1 The Schuster–Plummer sphere

We first consider the famous model by Schuster–Plummer (e.g., Ogorodnikov 1965) as the simplest model of a cluster. The potential and density of such a sphere are equal to

$$\Phi_N(r) = \frac{GM}{\sqrt{a^2+r^2}}\,, \qquad \text{and} \qquad \varrho(r) = \frac{3Ma^3}{4\pi(a^2+r^2)^{5/2}}$$

respectively. Here M is the mass of the model and a, the scale parameter. Hence

$$\frac{\mathrm{d}\Phi_N}{\mathrm{d}r} = -\frac{GMr}{(a^2+r^2)^{3/2}}\,, \qquad \varrho(0) = \frac{3M}{4\pi a^3}\,, \qquad \varrho(a) = \frac{3M}{16\sqrt{2}\pi a^3}\,.$$

Denote $M_\Lambda = \dfrac{4}{3}\,\pi a^3 \varrho_\Lambda$. The zero-gravity radius satisfies the following equation:

$$Ma^3 = 2M_\Lambda (a^2 + r_c^2)^{3/2}.$$

Then

$$r_c = \left[\left(\frac{M}{2M_\Lambda}\right)^{1/3} - 1\right] a, \qquad \text{or} \qquad r_c = \left[\left(\frac{\varrho(0)}{2\varrho_\Lambda}\right)^{1/3} - 1\right] a.$$

We see that $M = 2M_\Lambda$ if $r_c = 0$. Hence, circular orbits exist nearby the centre of the system only if $\varrho(0) > 2\varrho_\Lambda$, the effective background density must be low enough. Let $r_c > a$. Then $\varrho(0) > 16\varrho_\Lambda$.

The squared epicyclic frequency is equal to

$$\omega_r^2 = \frac{GM(4a^2+r^2)}{(a^2+r^2)^{5/2}} - \frac{32\pi}{3}\,G\varrho_\Lambda.$$

Let $x^{1/2} = r_*/a$ be the dimensionless critical radius. After some elementary algebra it can be found that

$$\frac{\varrho_\Lambda}{\varrho(a)} = \frac{4+x}{\sqrt{2}(1+x)^{5/2}}$$

(we compare ϱ_Λ with $\varrho(a)$ and not with $\varrho(0)$ because the model below has infinite central density).

One can see that for any x (i.e. for any distance from the centre) there exists such a minimal value $\varrho^*(x)$ that circular orbits are unstable at $\varrho_\Lambda > \varrho^*(x)$ (if they exist). All circular orbits are unstable if $\varrho_\Lambda > \varrho^*(0) > (4/\sqrt{2})\varrho(a)$. Thus there exists a critical minimal cluster density.

3.2 The isochrone model

We now consider the isochrone model by Hénon (1959) with the potential

$$\Phi_N(r) = \frac{GM}{a+\sqrt{a^2+r^2}}$$

and density

$$\varrho(r) = \frac{Ma\left[3a(a+\sqrt{a^2+r^2})+2r^2\right]}{4\pi\left[a+\sqrt{a^2+r^2}\right]^3(a^2+r^2)^{3/2}}\,.$$

Hence

$$\varrho(0) = \frac{3M}{16\pi a^3}\,, \qquad \varrho(a) = \frac{8-5\sqrt{2}}{16\pi}\,\frac{M}{a^3}\,.$$

We find that squared circular velocity

$$\omega^2 = \frac{GMr}{(a^2+r^2)^{1/2}(a+\sqrt{a^2+r^2})^2} - \frac{8\pi}{3}\,G\varrho_\Lambda.$$

Denote $z = (a^2+r_c^2)^{1/2}/a$, $y = M/(2M_\Lambda)$. Then

$$z(1+z)^2 = y.$$

Circular orbits exist when $z > 1$, $y > 4$, i.e. $\varrho(0) > 2\varrho_\Lambda$. Now we set $r_c = a$ (i.e. $z = \sqrt{2}$), then $y = \sqrt{2}(1+\sqrt{2})^2 \approx 8.24$. If Newtonian gravitation dominates, $y \gg 1$, we see that $r_c \approx y^{1/3}a$.

We find that

$$\omega_r^2 = GM\frac{\sqrt{a^2+r^2}(4a^2+r^2)+a(4a^2+3r^2)}{(a^2+r^2)^{3/2}(\sqrt{a^2+r^2}+a)^3} - \frac{32\pi}{3}\,G\varrho_\Lambda.$$

Given that $a^2 = (8-5\sqrt{2})M/[16\pi\varrho(a)]$ it follows that $\omega_r^2 = 0$ if

$$\frac{\varrho_\Lambda}{\varrho(a)} = \frac{3[\sqrt{1+x}(4+x)+4+3x]}{2(8-5\sqrt{2})(1+x+\sqrt{1+x})^3)}\,.$$

Here again $x = (r_*/a)^2$. We see that the maximal ϱ_Λ for which stable circular orbits exist is equal to $[6/(8-5\sqrt{2})]\varrho(a) \approx 7.23\varrho(a)$.

3.3 The 'limiting' model

We finally consider the model by Hernquist (1990). Kuzmin & Veltmann (1972) discovered it even earlier as the limiting case of a family of generalized isochrone models. This model is known to approximate the de Vaucouleurs profile. The 'limiting' model is sometimes used for representing the results of ΛCDM simulations. Its potential and density are equal to

$$\Phi_N(r) = \frac{GM}{a+r}\,, \qquad \varrho(r) = \frac{Ma}{2\pi\,r(a+r)^3}\,.$$

We see that $\varrho(r) \to \infty$ when $r \to 0$, and

$$\varrho(a) = \frac{M}{16\pi a^3}.$$

The zero-gravity radius r_c is a root of the following equation:

$$\frac{GM}{r_c(a+r_c)^2} = \frac{2GM_\Lambda}{a^3}\,.$$

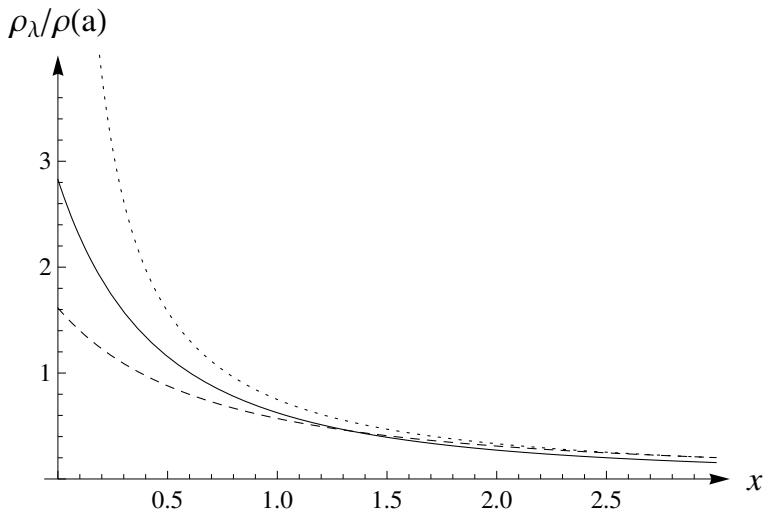


Figure 1 Plots of $\varrho_\Lambda/\varrho(a)$ as a function of x for the Schuster–Plummer (the solid line), the isochrone (the dashed line), and the 'limiting' (the dotted line) models.

Denoting $u = r_c/a$, $y = M/(2M_\Lambda)$, one can find that

$$u(1+u)^2 = y.$$

It is interesting that now $u = 0$ only when $y = 0$, i.e. circular orbits near the centre exist for any $M > 0$. We see that $y = 4$ for $u = 1$. The latter means that circular orbits exist for $r_c > a$ when $\varrho(a) > \dfrac{2}{3}\,\varrho_\Lambda$.

Now the epicyclic frequency is

$$\omega_r = \left[\frac{GM(3a+r)}{(a+r)^3 r} - \frac{32\pi}{3}\,G\varrho_\Lambda\right]^{1/2}.$$

The condition $\omega_r(r_*) = 0$ yields that

$$\frac{\varrho_\Lambda}{\varrho(a)} = \frac{3(3+\sqrt{x})}{2\sqrt{x}(1+\sqrt{x})^3}$$

($x = r_*/a$). We see that $\varrho_\Lambda/\varrho(a) = O(x^{3/2})$ for large x as for the isochrone model, but it should be remembered that any analysis at r greater than the zero-gravity radius r_c has no physical sense. For any ϱ_Λ circular orbits near the centre are stable.

4 Conclusion

We calculated the zero-gravity radius r_c and the critical radius r_* for the Schuster–Plummer model, isochrone model by Hénon, and the model by Kuzmin–Veltmann–Hernquist. We naturally found this radius to be greater for systems with greater ϱ_Λ.

In the case of the first two models there exists such a critical density that no stable circular orbits are possible in systems with densities below this threshold. The similar analysis for an arbitrary spherical model that is uniform at the centre is possible.

We would expect this instability to be one of the mechanisms of the preference of radial (but, of course, not purely radial) orbits in such systems, especially at their periphery. Radial flows are probable at $r > r_*$.

It would be interesting to estimate r_c and r_* for real clusters of galaxies. To do this, one has to know the mass distribution law for clusters (distribution of dark matter). It can be derived from the hydrostatic Courtez–Ogorodnikov equation (Coutrez 1950; Ogorodnikov 1957) using kinematic data or the temperature distribution from X-ray observations.

Further studies of the dynamics of systems in the presence of antigravity must take into account non-steady phenomena such as radial expansion.

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