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Investigation of the three-body-points problem with variable masses using the analytical calculation system "Mathematica": Theory

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The equations of motion of the investigated problem are obtained in the osculating elements of an aperiodic motion at the quasiconic section. Secular parts of the perturbing functions are calculated in terms of the osculating canonical elements with the use of the system "Mathematica". Full first-order secular perturbations are obtained by the method of Picard.

Keywords: Three-body problem; Variable masses; Secular perturbations; Computer algebra system "Mathematica"

1 Introduction

Real celestial bodies are non-stationary. Their masses, sizes, shapes and structure of the mass distribution within the body change in time (Omarov 1975; 2002; Bekov & Omarov 2003; Minglibayev 2012). These processes take place in binary and multiple systems (Lukyanov 2008). In this connection we investigate the three-body problem with masses varying isotropically at different rates. The bodies are considered as material points. Secular perturbations in the corresponding three-body problem with variable masses are investigated with the use of perturbation theory based on aperiodic motion at the quasiconic section (Minglibayev 1992).

2 Problem statement

Let us consider a system of three mutually gravitating bodies T_0, T_1 and T_2 with variable masses, changing isotropically at different specific rates (Omarov 1975)

$$m_0 = m_0(t), \quad m_1 = m_1(t), \quad m_2 = m_2(t), \quad \frac{\dot{m}_s}{m_s} \neq \frac{\dot{m}_\sigma}{m_\sigma}, \quad s, \sigma = 0, 1, 2, \quad s \neq \sigma. \tag{1}$$

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The equations of motion in an inertial Cartesian coordinate system have the form

$$m_i\ddot{\vec{R}}_i^* = grad_{\vec{R}_i^*}U, \quad i=0,1,2 \tag{2}$$

where

$$U=f\left(\frac{m_0m_1}{R_{01}^*}+\frac{m_0m_2}{R_{02}^*}+\frac{m_1m_2}{R_{12}^*}\right), \quad R_{ij}^*=\sqrt{(X_j^*-X_i^*)^2+(Y_j^*-Y_i^*)^2+(Z_j^*-Z_i^*)^2},$$

$\vec{R}_i^* = \vec{R}_i^*(X_i^*, Y_i^*, Z_i^*)$ are the radius vectors of the bodies, f is the gravitational constant. The system of equations (**??**) by the conditions (**??**) in contrast to other statements of the three-body problem with variable masses has no integral. Taking into account that the force function depends only on the mutual distances between the bodies, following (Omarov 1975), we pass to a relative coordinate system with the origin at the point T_0 with mass $m_0 = m_0(t)$. Let us denote

$$\vec{R}_i = \vec{R}_i(X_i, Y_i, Z_i) = \vec{R}_i^* - \vec{R}_0^*, \quad i=1,2. \tag{3}$$

The equations of relative motion have a classic form

$$\ddot{\vec{R}}_i + f(m_0+m_i)\frac{\vec{R}_i}{R_i^3} = grad_{\vec{R}_i}W_i, \tag{4}$$

$$W_i = f\sum_{j=1}^{2} m_j\left(\frac{1}{\Delta_{ij}} - \frac{X_iX_j+Y_iY_j+Z_iZ_j}{R_j^3}\right),$$

$$\Delta_{ij}=\sqrt{(X_j-X_i)^2+(Y_j-Y_i)^2+(Z_j-Z_i)^2}=R_{ij}^*,$$

$$\Delta_{i0}=R_i=\sqrt{X_i^2+Y_i^2+Z_i^2}.$$

If we know the solution of the differential equation (**??**), then the vector $\vec{R}_0^*$ is determined by integrating the equation

$$\ddot{\vec{R}}_0^* = grad_{\vec{R}_0^*}\left[f\left(\frac{m_1}{R_{01}^*}+\frac{m_2}{R_{02}^*}\right)\right],$$

and the vectors $\vec{R}_1^*, \vec{R}_2^*$ are easily determined from (**??**).

Using the equations of relative motion (**??**) we obtain the equations of motion in the Jacobi coordinates (Minglibayev 2012; Charlier 1966)

$$\mu_1\ddot{\vec{r}}_1 = grad_{\vec{r}_1}U, \quad \mu_2\ddot{\vec{r}}_2 = grad_{\vec{r}_2}U - \mu_2(2\dot{\nu}_1\dot{\vec{r}}_1+\ddot{\nu}_1\vec{r}_1), \tag{5}$$

where

$$\mu_1=\mu_1(t)=\frac{m_1m_0}{m_0+m_1}\neq const, \quad \mu_2=\mu_2(t)=\frac{m_2(m_1+m_0)}{m_0+m_1+m_2}\neq const$$

are the reduced masses,

$$U=f\left(\frac{m_0m_1}{r_{01}}+\frac{m_0m_2}{r_{02}}+\frac{m_1m_2}{r_{12}}\right),$$

$$r_{01}^2 = x_1^2 + y_1^2 + z_1^2 = r_1^2, \quad r_2^2 = x_2^2 + y_2^2 + z_2^2,$$

$$r_{02}^2 = (x_2 + \nu_1 x_1)^2 + (y_2 + \nu_1 y_1)^2 + (z_2 + \nu_1 z_1)^2,$$

$$r_{12}^2 = (x_2 - \nu_0 x_1)^2 + (y_2 - \nu_0 y_1)^2 + (z_2 - \nu_0 z_1)^2,$$

$$\nu_1 = \nu_1(t) = \frac{m_1}{m_0 + m_1} \neq const, \quad \nu_0 = \nu_0(t) = \frac{m_0}{m_0 + m_1} \neq const.$$

We investigate the equations of motion (**??**) using the perturbation theory based on aperiodic motion at the quasiconic section (Minglibayev 1992).

3 The equations of motion in osculating elements

We rewrite equation (**??**) in the form

$$\mu_1 \ddot{\vec{r}}_1 = grad_{\vec{r}_1} \left(f \frac{m_1 m_0}{r_1} \right) + b_1 \vec{r}_1 + grad_{\vec{r}_1} R_1, \tag{6}$$

$$\mu_2 \ddot{\vec{r}}_2 = grad_{\vec{r}_2} \left(f \frac{m_2(m_1 + m_0)}{r_2} \right) + b_2 \vec{r}_2 + grad_{\vec{r}_2} R_2, \tag{7}$$

where

$$b_1 = b_1(t) = \mu_1 \frac{\ddot{\gamma}_1}{\gamma_1}, \quad \gamma_1 = \gamma_1(t) = \frac{m_0(t_0) + m_1(t_0)}{m_0(t) + m_1(t)},$$

$$b_2 = b_2(t) = \mu_2 \frac{\ddot{\gamma}_2}{\gamma_2}, \quad \gamma_2 = \gamma_2(t) = \frac{m_0(t_0) + m_1(t_0) + m_2(t_0)}{m_0(t) + m_1(t) + m_2(t)},$$

$$R_1 = -\frac{1}{2} b_1 r_1^2 + W, \quad R_2 = -\frac{1}{2} b_2 r_2^2 + W - V,$$

$$W = f \left(\frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}} - \frac{m_2(m_0 + m_1)}{r_2} \right),$$

$$V = \mu_2 [(2\dot{\nu}_1 \dot{x}_1 + \ddot{\nu}_1 x_1) x_2 + (2\dot{\nu}_1 \dot{y}_1 + \ddot{\nu}_1 y_1) y_2 + (2\dot{\nu}_1 \dot{z}_1 + \ddot{\nu}_1 z_1) z_2].$$

Let us introduce the following designations

$$K_1 = \frac{1}{2} \mu_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2), \quad K_2 = \frac{1}{2} \mu_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2),$$

$$U_1 = f \frac{m_1 m_0}{\sqrt{x_1^2 + y_1^2 + z_1^2}} + \frac{1}{2} b_1 (x_1^2 + y_1^2 + z_1^2) + R_1,$$

$$U_2 = f \frac{m_2(m_0 + m_1)}{\sqrt{x_2^2 + y_2^2 + z_2^2}} + \frac{1}{2} b_2 (x_2^2 + y_2^2 + z_2^2) + R_2,$$

and transfer to new variables

$$x_i = \gamma_i \rho_i \cos\varphi_i \cos\theta_1, \; y_i = \gamma_i \rho_i \cos\varphi_i \sin\theta_i, \quad z_i = \gamma_i \rho_i \sin\varphi_i, \; i = 1, 2.$$

The equations of motion in the semi-canonical variables

$$\rho_i, \quad P_{\rho_i} = \frac{\partial K_i}{\partial \dot{\rho}_i} = \mu_i \gamma_i^2 \dot{\rho}_i + \mu_i \gamma_i \dot{\gamma}_i \rho_i,$$
$$\varphi_i, \quad P_{\varphi_i} = \frac{\partial K_i}{\partial \dot{\varphi}_i} = \mu_i \gamma_i^2 \rho_i^2 \dot{\varphi}_i,$$
$$\theta_i, \quad P_{\theta_i} = \frac{\partial K_i}{\partial \dot{\theta}_i} = \mu_i \gamma_i^2 \rho_i^2 \cos^2 \varphi_i \dot{\theta}_i, \quad i = 1, 2,$$

may be written in the form

$$\dot{\rho}_i = \frac{\partial H_i}{\partial P_{\rho_i}}, \quad \dot{P}_{\rho_i} = -\frac{\partial H_i}{\partial \rho_i} + \frac{\dot{\mu}_i}{\mu_i} P_{\rho_i},$$
$$\dot{\varphi}_i = \frac{\partial H_i}{\partial P_{\varphi_i}}, \quad \dot{P}_{\varphi_i} = -\frac{\partial H_i}{\partial \varphi_i} + \frac{\dot{\mu}_i}{\mu_i} P_{\varphi_i},$$
$$\dot{\theta}_i = \frac{\partial H_i}{\partial P_{\theta_i}}, \quad \dot{P}_{\theta_i} = -\frac{\partial H_i}{\partial \theta_i} + \frac{\dot{\mu}_i}{\mu_i} P_{\theta_i},$$

where

$$H_i = \frac{1}{2\mu_i \gamma_i^2} \left[\left(P_{\rho_i} - \mu_i \gamma_i \dot{\gamma}_i \rho_i \right)^2 + \frac{P_{\varphi_i}^2}{\rho_i^2} + \frac{P_{\theta_i}^2}{\rho_i^2 \cos^2 \varphi_i} \right] - U_i^*,$$
$$U_1^* = f \frac{m_1 m_0}{\gamma_1 \rho_1} + \frac{1}{2} (b_1 \gamma_1^2 + \mu_1 \dot{\gamma}_1^2) \rho_1^2 + R_1,$$
$$U_2^* = f \frac{m_2 (m_0 + m_1)}{\gamma_2 \rho_2} + \frac{1}{2} (b_2 \gamma_2^2 + \mu_2 \dot{\gamma}_2^2) \rho_2^2 + R_2.$$

Introducing the new impulses

$$P_{\rho_i} = \psi_i \tilde{P}_{\rho_i}, \quad P_{\varphi_i} = \psi_i \tilde{P}_{\varphi_i}, \quad P_{\theta_i} = \psi_i \tilde{P}_{\theta_i},$$
$$\psi_i = \psi_i(t) = \frac{\mu_i}{\mu_{i0}} = \frac{\mu_i(t)}{\mu_i(t_0)},$$

the motion equations may be written in the canonical form

$$\begin{aligned} \dot{\rho}_i &= \frac{\partial \tilde{H}_i}{\partial \tilde{P}_{\rho_i}}, \qquad \dot{\varphi}_i = \frac{\partial \tilde{H}_i}{\partial \tilde{P}_{\varphi_i}}, \qquad \dot{\theta}_i = \frac{\partial \tilde{H}_i}{\partial \tilde{P}_{\theta_i}}, \\ \dot{\tilde{P}}_{\rho_i} &= -\frac{\partial \tilde{H}_i}{\partial \rho_i}, \quad \dot{\tilde{P}}_{\varphi_i} = -\frac{\partial \tilde{H}_i}{\partial \varphi_i}, \quad \dot{\tilde{P}}_{\theta_i} = -\frac{\partial \tilde{H}_i}{\partial \theta_i}, \end{aligned} \tag{8}$$

where

$$\tilde{H}_i = \tilde{H}_{i0} + \tilde{H}_{i1}, \tag{9}$$

$$\begin{aligned} \tilde{H}_{10} &= \frac{\psi_1}{2\mu_1 \gamma_1^2} \left[\left(\tilde{P}_{\rho_1} - \frac{\mu_1 \gamma_1 \dot{\gamma}_1}{\psi_1} \rho_1 \right)^2 + \frac{\tilde{P}_{\varphi_1}^2}{\rho_1^2} + \right. \\ &\left. + \frac{\tilde{P}_{\theta_1}^2}{\rho_1^2 \cos^2 \varphi_1} \right] - f \frac{m_1 m_0}{\psi_1 \gamma_1 \rho_1} - \frac{1}{\psi_1} (b_1 \gamma_1^2 + \mu_1 \dot{\gamma}_1^2) \rho_1^2, \\ \tilde{H}_{20} &= \frac{\psi_2}{2\mu_2 \gamma_2^2} \left[\left(\tilde{P}_{\rho_2} - \frac{\mu_2 \gamma_2 \dot{\gamma}_2}{\psi_2} \rho_2 \right)^2 + \frac{\tilde{P}_{\varphi_2}^2}{\rho_2^2} + \frac{\tilde{P}_{\theta_2}^2}{\rho_2^2 \cos^2 \varphi_2} \right] - \\ &- f \frac{m_2 (m_0 + m_1)}{\psi_2 \gamma_2 \rho_2} - \frac{1}{\psi_2} (b_2 \gamma_2^2 + \mu_2 \dot{\gamma}_2^2) \rho_2^2, \end{aligned} \tag{10}$$

$$\tilde{H}_{11} = -\frac{1}{\psi_1} R_1, \quad \tilde{H}_{21} = -\frac{1}{\psi_2} R_2. \tag{11}$$

If $R_1 = R_2 = 0$, equations (**??**)–(**??**) define the unperturbed motion and can be integrated by the Hamilton–Jacobi method (Minglibayev 1992). The integration constants

$$\alpha_{1i}, \quad \alpha_{2i}, \quad \alpha_{3i}, \quad \beta_{1i}, \quad \beta_{2i}, \quad \beta_{3i}, \quad i = 1, 2, \tag{12}$$

are the analogues of the Jacobi elements in the classical problem of two bodies with constant masses. When ($H_{i1} = 0$, $R_i = 0$), each system of equations (**??**)–(**??**) defines the aperiodic unperturbed motion at the quasiconic section (Minglibayev 2012)

$$r_i = \gamma_i(t)\rho_i, \;\; \rho_i = \frac{p_i}{1 + e_i \cos \upsilon_i}, \;\; \upsilon_i = u_i - \omega_i, \;\; i = 1, 2,$$

$$p_i = a_i(1 - e_i^2), \quad \dot{\rho}_i = \dot{\rho}_i(t) = \frac{1}{\mu_{i0}\gamma_i^2(t)} \cdot \frac{\tilde{\beta}_i}{\sqrt{p_i}} e_i \sin \upsilon_i, \quad \dot{u}_i = \frac{1}{\mu_{i0}\gamma_i^2(t)} \cdot \frac{\tilde{\beta}_i \sqrt{p_i}}{\rho_i^2},$$

$$\tilde{\beta}_1^2 = f \cdot \mu_1(t_0) m_1(t_0) m_0(t_0), \quad \tilde{\beta}_2^2 = f \cdot \mu_2(t_0) m_2(t_0)[m_0(t_0) + m_1(t_0)],$$

$$tg\frac{\upsilon_i}{2} = \frac{\sqrt{1 + e_i}}{\sqrt{1 - e_i}} tg\frac{E_i}{2}, \quad e_i < 1, \quad E_i - e_i \sin E_i = M_i,$$

$$M_i = n_i \left[\phi_i(t) - \phi_i(\tau_i)\right], \;\; n_i = \frac{\tilde{\beta}_i}{\mu_{i0} a_i^{3/2}}, \;\; \mu_{i0} = \mu_i(t_0),$$

where $\phi_1(t)$, $\phi_2(t)$ are the primitive functions of $\gamma_1^{-2}(t)$ and $\gamma_2^{-2}(t)$ correspondingly. The values

$$a_i, \quad e_i, \quad \omega_i, \quad \Omega_i, \quad i_i, \quad \phi_i(\tau_i) \tag{13}$$

are the analogues of the Kepler elements and are connected with expressions (**??**) according to the formulas

$$\begin{array}{lll} -2\alpha_{1i} = \dfrac{\tilde{\beta}_i^2}{\mu_{i0} a_i}, & \alpha_{2i} = \tilde{\beta}_i \sqrt{p_i}, & \alpha_{3i} = \tilde{\beta}_i \sqrt{p_i} \cos i_i, \\ \beta_{1i} = -\phi_i(\tau_i), & \beta_{2i} = \omega_i, & \beta_{3i} = \Omega_i. \end{array} \tag{14}$$

Coordinates and velocities of the two bodies with reduced masses $\mu_1(t), \mu_2(t)$ in the system of Jacobi coordinates are given by

$$\begin{aligned} x_i &= \gamma_i \rho_i \left[\cos u_i \cdot \cos \Omega_i - \sin u_i \cdot \sin \Omega_i \cdot \cos i_i\right], \\ y_i &= \gamma_i \rho_i \left[\cos u_i \cdot \sin \Omega_i + \sin u_i \cdot \cos \Omega_i \cdot \cos i_i\right], \\ z_i &= \gamma_i \rho_i \left[\sin u_i \cdot \sin i_i\right], \quad r_i^2 = x_i^2 + y_i^2 + z_i^2, \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{x}_i &= \left(\frac{\dot{\gamma}_i}{\gamma_i} + \frac{\dot{\rho}_i}{\rho_i}\right) x_i + \gamma_i \rho_i \dot{u}_i \cdot \left[-\sin u_i \cdot \cos \Omega_i - \cos u_i \cdot \sin \Omega_i \cdot \cos i_i\right], \\ \dot{y}_i &= \left(\frac{\dot{\gamma}_i}{\gamma_i} + \frac{\dot{\rho}_i}{\rho_i}\right) y_i + \gamma_i \rho_i \dot{u}_i \cdot \left[-\sin u_i \cdot \sin \Omega_i + \cos u_i \cdot \cos \Omega_i \cdot \cos i_i\right], \\ \dot{z}_i &= \left(\frac{\dot{\gamma}_i}{\gamma_i} + \frac{\dot{\rho}_i}{\rho_i}\right) z_i + \gamma_i \rho_i \dot{u}_i \cdot \left[\cos u_i \cdot \sin i_i\right]. \end{aligned}$$

When $R_1 \neq 0\,(H_{11} \neq 0)$, $R_2 \neq 0\,(H_{12} \neq 0)$, equations (**??**)–(**??**) as equations of perturbed motion in variables (**??**) may be written in the form

$$\dot{\alpha}_{1k} = \frac{\partial \tilde{R}_1}{\partial \beta_{1k}}, \quad \dot{\beta}_{1k} = -\frac{\partial \tilde{R}_1}{\partial \alpha_{1k}},$$
$$\dot{\alpha}_{2k} = \frac{\partial \tilde{R}_2}{\partial \beta_{2k}}, \quad \dot{\beta}_{2k} = -\frac{\partial \tilde{R}_2}{\partial \alpha_{2k}}, \quad k = 1, 2, 3,$$

where

$$\tilde{R}_i = \frac{1}{\psi_i} R_i(t, \alpha_{11}, \alpha_{12}, \alpha_{13}, \beta_{11}, \beta_{12}, \beta_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \beta_{21}, \beta_{22}, \beta_{23}), \quad i = 1, 2.$$

Then it is possible to enter different canonical variables, such as the analogues of the canonical Delaunay elements

$$L_i, \quad G_i, \quad H_i, \quad l_i, \quad g_i, \quad h_i, \quad i = 1, 2, \tag{16}$$

by the formulas

$$\begin{aligned} -2\alpha_{1i} &= \frac{\tilde{\beta}_i^4}{\mu_{i0} L_i^2}, \quad \alpha_{2i} = G_i, \quad \alpha_{3i} = H_i, \\ \beta_{1i} &= \frac{l_i}{n_i} - \phi_i(t), \quad \beta_{2i} = g_i, \quad \beta_{3i} = h_i. \end{aligned} \tag{17}$$

It should be noted that we have introduced analogues of the Delaunay elements (**??**), which are quite different from the "elements like Delaunay" introduced by Deprit (1983) and developed further in the work of Floria (1997). For our investigation the analogues of the Poincaré second system are more convenient.

4 The perturbed equations of motion in the analogues of the second system of the Poincaré elements

Analogues of the second system of the Poincaré elements (Minglibayev 2012)

$$\Lambda_i, \quad \lambda_i, \quad \xi_i, \quad \eta_i, \quad p_i, \quad q_i, \tag{18}$$

are defined as follows

$$\begin{aligned} \Lambda_i &= \tilde{\beta}_i \sqrt{a_i}, \\ \lambda_i &= n_i[\phi(t) - \phi(\tau)] + \pi_i = l_i + \Omega_i + \omega_i, \\ \xi_i &= \sqrt{2\tilde{\beta}_i \sqrt{a_i}(1 - \sqrt{1 - e_i^2})} \cos \pi_i, \\ \eta_i &= -\sqrt{2\tilde{\beta}_i \sqrt{a_i}(1 - \sqrt{1 - e_i^2})} \sin \pi_i, \\ p_i &= \sqrt{2\tilde{\beta}_i \sqrt{a_i(1 - e_i^2)}(1 - \cos i_i)} \cos \Omega_i, \\ q_i &= -\sqrt{2\tilde{\beta}_i \sqrt{a_i(1 - e_i^2)}(1 - \cos i_i)} \sin \Omega_i. \end{aligned} \tag{19}$$

According to formulas (**??**), (**??**) the Poincaré elements are connected with the aperiodic motion at the quasiconic section (**??**)–(**??**).

Equations of perturbed motion have the form

$$\dot{\Lambda}_i = \frac{\partial R_i^*}{\partial \lambda_i}, \qquad \dot{\xi}_i = \frac{\partial R_i^*}{\partial \eta_i}, \qquad \dot{p}_i = \frac{\partial R_i^*}{\partial q_i},$$
$$\dot{\lambda}_i = -\frac{\partial R_i^*}{\partial \Lambda_i}, \quad \dot{\eta}_i = -\frac{\partial R_i^*}{\partial \xi_i}, \quad \dot{q}_i = -\frac{\partial R_i^*}{\partial p_i},$$

Considering the non-resonant case and averaging the perturbation functions by λ_i, we can obtain the equations of secular perturbations

$$\begin{gathered}\dot{\Lambda}_1 = 0, \quad \dot{\Lambda}_2 = 0,\\ \dot{\xi}_i = \frac{\partial R_{i\mathrm{sec}}}{\partial \eta_i}, \qquad \dot{p}_i = \frac{\partial R_{i\mathrm{sec}}}{\partial q_i},\\ \dot{\eta}_i = -\frac{\partial R_{i\mathrm{sec}}}{\partial \xi_i}, \quad \dot{q}_i = -\frac{\partial R_{i\mathrm{sec}}}{\partial p_i},\end{gathered} \tag{20}$$

where $R_{1\mathrm{sec}}$, $R_{2\mathrm{sec}}$ are the corresponding secular part of the following expressions

$$R_1^* = \frac{1}{\gamma_1^2(t)} \cdot \frac{\tilde{\beta}_1^4}{2\mu_{10}\Lambda_1^2} + \frac{1}{\psi_1}\left[-\frac{b_1}{2}\gamma_1^2\rho_1^2 + f\left(\frac{m_0m_2}{r_{02}} + \frac{m_1m_2}{r_{12}} - \frac{m_2(m_0+m_1)}{\gamma_2\rho_2}\right)\right],$$

$$R_2^* = \frac{1}{\gamma_2^2(t)} \cdot \frac{\tilde{\beta}_2^4}{2\mu_{20}\Lambda_2^2} + \frac{1}{\psi_2}\left[-\frac{b_2}{2}\gamma_2^2\rho_2^2 + f\left(\frac{m_0m_2}{r_{02}} + \frac{m_1m_2}{r_{12}} - \frac{m_2(m_0+m_1)}{\gamma_2\rho_2}\right)\right] -$$
$$-\frac{\mu_2}{\psi_2}\left[(2\dot{\nu}_1\dot{x}_1 + \ddot{\nu}_1x_1)x_2 + (2\dot{\nu}_1\dot{y}_1 + \ddot{\nu}_1y_1)y_2 + (2\dot{\nu}_1\dot{z}_1 + \ddot{\nu}_1z_1)z_2\right].$$

5 Expansion of the perturbation function

To calculate the secular parts of the perturbed functions it is necessary to obtain the secular part of the following expressions

$$F_{\mathrm{sec}} = \left[\frac{m_0m_2}{r_{02}} + \frac{m_1m_2}{r_{12}} - \frac{m_2(m_0+m_1)}{\gamma_2\rho_2}\right]_{\mathrm{sec}} = \left[\frac{m_1m_2}{r_{12}}\right]_{\mathrm{sec}},$$

$$F_{\rho\mathrm{sec}} = \left[\frac{b_1\gamma_1^2}{2\psi_1}\rho_1^2 + \frac{b_2\gamma_2^2}{2\psi_2}\rho_2^2\right]_{\mathrm{sec}},$$

$$V_{\mathrm{sec}} = \frac{\mu_2}{\psi_2}[(2\dot{\nu}_1\dot{x}_1 + \ddot{\nu}_1x_1)x_2 + (2\dot{\nu}_1\dot{y}_1 + \ddot{\nu}_1y_1)y_2 + (2\dot{\nu}_1\dot{z}_1 + \ddot{\nu}_1z_1)z_2]_{\mathrm{sec}}.$$

Let us assume that the elements e_i, i_i are sufficiently small. Therefore, we can expand the perturbation function in a series by small parameters e_i, i_i and take into account only terms including the second order terms. In contrast to the two planetary three-body problem we do not assume the masses $m_1(t)$, $m_2(t)$ to be small.

In the analogues of the second system of the Poincaré elements the secular expressions for $R_{1\mathrm{sec}}$, $R_{2\mathrm{sec}}$ have the form

$$R_{1\mathrm{sec}}=\frac{1}{\gamma_1^2(t)}\cdot\frac{\tilde{\beta}_1^4}{2\mu_{10}\Lambda_1^2}+\frac{1}{\psi_1}\left[-\frac{b_1}{2}\frac{\gamma_1^2\mu_0^2}{\Lambda_1^4}\left(1+\frac{3}{2}\frac{\xi_1^2+\eta_1^2}{\Lambda_1}\right)+F_{\mathrm{sec}}\right], \tag{21}$$

$$R_{2\mathrm{sec}}=\frac{1}{\gamma_2^2(t)}\cdot\frac{\tilde{\beta}_2^4}{2\mu_{20}\Lambda_2^2}+\frac{1}{\psi_2}\left[-\frac{b_2}{2}\frac{\gamma_2^2\mu_0^2}{\Lambda_2^2}\left(1+\frac{3}{2}\frac{\xi_2^2+\eta_2^2}{\Lambda_2}\right)+F_{\mathrm{sec}}\right]-V_{\mathrm{sec}}, \tag{22}$$

where

$$V_{\mathrm{sec}}=\frac{9\Lambda_1\Lambda_2\mu_2\gamma_2(p_1p_2+q_1q_2)(2\dot{\gamma}_1\dot{\nu}_1+\gamma_1\ddot{\nu}_1)}{16\mu_{10}\mu_{20}\psi_2}\times$$
$$\times\sqrt{p_1^2+q_1^2}\sqrt{p_2^2+q_2^2}\sqrt{(\eta_1^2+\xi_1^2)(\eta_2^2+\xi_2^2)}\sqrt{(4\Lambda_1-\eta_1^2-\xi_1^2)(4\Lambda_2-\eta_2^2-\xi_2^2)},$$

$$F_{\mathrm{sec}}=\left[\frac{m_1m_2}{r_{12}}\right]_{\mathrm{sec}}. \tag{23}$$

Calculation of F_{sec} is a separate, hard time-consuming problem. We have calculated formula (**??**) with the computer algebra system "Mathematica" (Prokopenya 2005). The obtained expressions are very cumbersome and difficult to foresee, therefore, we write the final expression in the form

$$F=\sum_{i=1}^{534}\Pi_i^*(t)P_i(\ni_k)+\sum_{j=1}^{3}\tilde{\Pi}_j(\Lambda_1,\Lambda_2,t).$$

where $\ni_k$=$\ni_k$ $(\xi_1,\eta_1,p_1,q_1,\xi_2,\eta_2,p_2,q_2)$.

6 Solutions of the secular perturbation equations

Substituting equations (**??**)–(**??**) in (**??**) we obtain

$$\begin{aligned}\dot{\xi}_1 &= \Pi_1(t)\eta_1+\Pi_2(t)\eta_2, & \dot{\eta}_1 &= \Pi_1(t)\xi_1+\Pi_2(t)\xi_2,\\ \dot{\xi}_2 &= \Pi_3(t)\eta_1+[\Pi_4(t)+\Phi_1(t,\ni_k)]\,\eta_2, & \dot{\eta}_2 &= \Pi_3(t)\xi_1+[\Pi_4(t)+\Phi_1(t,\ni_k)]\,\xi_2,\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{p}_1 &= \Pi_5(t)q_1+\Pi_6(t)q_2, & \dot{q}_1 &= \Pi_5(t)p_1+\Pi_6(t)p_2,\\ \dot{p}_2 &= \Pi_7(t)q_1+[\Pi_8(t)+\Phi_2(t,\ni_k)]\,q_2, & \dot{q}_2 &= \Pi_7(t)p_1+[\Pi_8(t)+\Phi_2(t,\ni_k)]\,p_2,\end{aligned} \tag{25}$$

where

$$\Pi_1(t)=\frac{-24\Lambda_2^4\mu_{10}^4\mu_1\gamma_1\ddot{\gamma}_1+12fk_1\Lambda_1^2\Lambda_1^4m_1m_2[k_1\Lambda_1^2C_0\mu_{20}^2-\Lambda_2^2\mu_{10}\mu_{20}(B_0+k_2C_1)]}{16\Lambda_1^5\Lambda_2^4\mu_{10}^2\psi_1}+$$
$$+\frac{fk_2m_1m_2[3k_2(10C_0-3C_2)-4B_1]}{16\Lambda_1\psi_1},$$

$$\Pi_2(t)=-\frac{6fk_2m_1m_2(k_1^3\Lambda_1^4\mu_{20}^2+k_2^2\Lambda_2^4\mu_{10}^2)(6C_0-C_2)}{32k_1^2\Lambda_1^{5/2}\Lambda_2^{5/2}\mu_{10}\mu_{20}\psi_1}+$$
$$+\frac{fk_2m_1m_2[2k_1(B_2+18B_0)+3k_2C_1(4+3k_1)]}{32k_1\Lambda_1^{1/2}\Lambda_2^{1/2}\psi_1},$$

$$\Pi_3(t)=-\frac{6fk_2m_1m_2(k_1^3\Lambda_1^4\mu_{20}^2+k_2^2\Lambda_2^4\mu_{10}^2)(6C_0-C_2)}{32k_1^2\Lambda_1^{5/2}\Lambda_2^{5/2}\mu_{10}\mu_{20}\psi_2}+$$
$$+\frac{fk_2m_1m_2[2k_1(B_2+18B_0)+3k_2C_1(4+3k_1)]}{32k_1\Lambda_1^{1/2}\Lambda_2^{1/2}\psi_2},$$
$$\Pi_4(t)=\frac{-24k_1^4\Lambda_1^4\mu_{20}^4\mu_2\gamma_2\ddot{\gamma}_2+12fk_2^2\Lambda_2^6m_1m_2[k_2^2\Lambda_2^2C_0\mu_{10}^2-k_1^2\Lambda_1^2\mu_{10}\mu_{20}(B_0+k_2C_1)]}{16\Lambda_1^5\Lambda_2^4k_1^4\mu_{20}^2\psi_2}-$$
$$-\frac{fk_2m_1m_2[4B_1+3k_2(3C_2-10C_0)]}{16\Lambda_1\psi_2},$$
$$\Pi_5(t)=-\frac{fk_2B_1m_1m_2}{8\Lambda_1\psi_1},\ \Pi_6(t)=\frac{fk_2B_1m_1m_2}{8\sqrt{\Lambda_1}\sqrt{\Lambda_2}\psi_1},$$
$$\Pi_7(t)=-\frac{fk_2B_1m_1m_2}{8\Lambda_2\psi_2},\ \Pi_8(t)=\frac{fk_2B_1m_1m_2}{8\sqrt{\Lambda_1}\sqrt{\Lambda_2}\psi_2},$$
$$\Phi_1(t,\ni_k)=\frac{2(\eta_2^2-2\Lambda_2+\xi_2^2)V_{\rm sec}}{(\eta_2^2+\xi_2^2)(4\Lambda_2-\eta_2^2-\xi_2^2)},\quad \Phi_2(t,\ni_k)=\frac{(p_1q_2-q_1p_2)V_{\rm sec}}{(p_2^2+q_2^2)(p_1p_2+q_1q_2)}.$$

The expressions

$$A_0=\frac{4\mu_0}{\Lambda_2^2\pi}F(\alpha),\quad A_1=\frac{4\mu_0}{\Lambda_2^2\alpha\pi}[F(\alpha)-E(\alpha)],\quad A_2=\frac{2}{3}\left(\alpha+\frac{1}{\alpha}\right)A_1-\frac{1}{3}A_0,$$
$$B_0=\frac{(1+\alpha^2)\alpha}{(1-\alpha^2)^2}A_0-\frac{2\alpha^2}{(1-\alpha^2)^2}A_1,\quad B_1=\frac{2\alpha^2}{(1-\alpha^2)^2}A_0-\frac{(1+\alpha^2)\alpha}{(1-\alpha^2)^2}A_1,$$
$$B_2=\frac{(1+\alpha^2)\alpha}{(1-\alpha^2)^2}A_0-\frac{2(1+\alpha^2)^2-6\alpha^2}{(1-\alpha^2)^2}A_1,$$
$$C_0=\frac{(3+10\alpha^2+3\alpha^4)\alpha^2}{(1-\alpha^2)^2}A_0-\frac{2\alpha^2(1+\alpha+\alpha^3)}{(1-\alpha^2)^2}A_1,$$
$$C_1=\frac{8\alpha^3(1+\alpha^2)}{3(1-\alpha^2)^4}A_0-\frac{\alpha^2(1+14\alpha^2+\alpha^4)}{3(1-\alpha^2)^4}A_1$$
$$C_2=\frac{(1+22\alpha^2+\alpha^4)\alpha^2}{3(1-\alpha^2)^4}A_0-\frac{2\alpha^2(1+5\alpha^2+5\alpha^4+\alpha^6)}{3(1-\alpha^2)^4}A_1,$$

are the Laplace coefficients (Charlier 1966), $F(\alpha)$ is an elliptic integral of the first kind, $E(\alpha)$ is an elliptic integral of the second kind, and the expressions

$$\alpha=\alpha(t)=\alpha_0\cdot\frac{m_{00}+m_{10}}{m_{00}+m_{10}+m_{20}}\cdot\frac{m_0(m_0+m_1+m_2)}{(m_0+m_1)(m_0+m_2)},\quad \alpha_0=\frac{a_{01}}{a_{02}}=const,$$
$$k_1=k_1(t)=\frac{m_0(m_{00}+m_{10})}{(m_0+m_1)(m_0+m_2)},\quad k_2=k_2(t)=k_1\frac{m_{00}+m_{10}+m_{20}}{m_0+m_1+m_2}.$$

Using the Picard method, we obtain solution of equations (**??**)–(**??**) in the form

$$\ni_k\ (t)=\ni_k\ (t_0)+\int_{t_0}^{t}\Pi_i^{**}(t,\ni_k\ (t_0))dt,\qquad (26)$$

where $\Pi_i^{**}(t,\ni_k)$ are the right-hand sides of these equations, $\ni_k$ are the elements ξ_i, η_i, p_i, q_i and $\ni_k\ (t_0)$ are their values at the initial time.

7 Conclusion

Solutions (**??**) allow us to analyze the evolution of the analogues of the eccentricities e_1, e_2, inclinations of the orbits i_1, i_2, arguments of the pericentre ω_1, ω_2 and the ascending nodes longitude Ω_1, Ω_2, longitudes of the pericentre π_1, π_2. Analytical expressions of these have the form

$$e_i^2 = \frac{\xi_i^2 + \eta_i^2}{\Lambda_i}, \quad \sin^2 i_i = \frac{p_i^2 + q_i^2}{\Lambda_i}, \quad i = 1, 2,$$

$$\Omega_i = -\arctan\frac{q_i}{p_i}, \quad \pi_i = -\arctan\frac{\eta_i}{\xi_i}, \quad \omega_i = \pi_i - \Omega_i.$$

The obtained approximate analytical general solutions allow us to analyze the problem analytically and numerically at the original absolute frame. In particular it is possible to study the motion of barycenter, which will be realized in the forthcoming paper.

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