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On the method for the analysis of compulsive phase mixing and its application in cosmogony

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In this paper, we study the strong non-stationary stochastic processes that take place in the phase space of self-gravitating systems at the earlier non-stationary stage of their evolution. The numerical calculations of the compulsive phase mixing process were carried out according to the model of chaotic impacts, where the initially selected phase volume experiences random pushes that are of a diverse and complex nature. The application of the method for studying random impacts on a volume element in the case of three-dimensional space is carried out.

Keywords: Phase mixing, violent relaxation, stochastic process, random impacts, self-gravitating system

1 Introduction

As it is known, at the initial non-stationary evolution stage of a collisionless self-gravitating system, phase mixing takes place which tends to bring the system closer to a quasistationary state in the regular field (Antonov et al. 1995; Binney and Tremaine 2008). The statistical mechanics of strongest non-stationary case was first studied by Lynden-Bell (1967), who called the corresponding physical process violent relaxation. After the classical work of Lynden-Bell (1967), a statistical study of violent relaxation became the main topic of discussion and the formation of star clusters, galaxies and their clusters were reviewed. However, by numerical experiments (Madsen 1987; Merritt et al. 1989) it turned out that the predicted final coarse-grained distribution takes place only under extremely non-stationary initial conditions and only in one-dimensional analysis of this process. It should be noted that the work by Antonov et al. (1995) proposed a classification of phase mixing depending on the degree of initial non-stationarity $(2T/|U|)_0$ of the system, where T and U are the components of the kinetic and potential energy of the system, respectively. Within the framework of this work, we study the stochastic evolution of a selected volume

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element in four dimensional phase space of a system with its most non-stationary state $(2T/|U|)_0 << 1$, which corresponds to compulsive phase mixing. The violent relaxation of Lynden-Bell (1967) is a special case of compulsive phase mixing. In the compulsive phase mixing process, especially in the case of violent relaxation, the total gravitational field of the system changes stochastically at any point of the space over time, and because of this, any small phase volume of the system will not be able to evolve independently. Thus, the phase volume is subjected to numberless random impacts (influences), which have different nature and intensity. We are interested in the statistical effect – the result of many influences. Calculations show that the initial random phase volume is statistically elongated on average in one direction, which confirms Lynden-Bell's predictions (Lynden-Bell 1967). Next, we are interested in this question in the case of a space with an odd dimension equal to 3. In this case, the statistical effect leads to the formation of a flat, almost disk-like structure. Our method can be applied, in principle, to some cosmogony processes (see, e.g., Chernin 1970; Peebles 1970).

2 Phase model of chaotic impacts

Let in the initial state there be an element of phase volume

$$\overrightarrow{q}^2 = 1, \qquad \overrightarrow{q} = \overrightarrow{q}\,(\overrightarrow{r}, \overrightarrow{v})\,. \tag{1}$$

If at moment t_1 a random impact on a selected element of the phase volume is described by a random matrix P_1, then after the first impact we have $\mathrm{P}_1\overrightarrow{q} = \overrightarrow{q}_1$. From here we find $\overrightarrow{q} = \mathrm{P}_1^{-1}\overrightarrow{q}_1$ and substitute in (1). We get

$$\overrightarrow{q}_1^{\,T}\left(\mathrm{P}_1\mathrm{P}_1^T\right)^{-1}\overrightarrow{q}_1 = 1, \tag{2}$$

where the symbol T means transposition of the matrix. Expression (2) is the equation of an ellipsoid whose semi-axes are determined by the eigenvalues of a random matrix $\mathrm{P}_1\mathrm{P}_1^T$. Thus, after the next impact in (2) we add the effect of the second "push"

$$\overrightarrow{q}_2^{\,T}\left(\mathrm{P}_2\mathrm{P}_1\mathrm{P}_1^T\mathrm{P}_2^T\right)^{-1}\overrightarrow{q}_2 = 1, \tag{3}$$

where $\overrightarrow{q}_2 = \mathrm{P}_2\overrightarrow{q}_1$ and $\overrightarrow{q}_1 = \mathrm{P}_2^{-1}\overrightarrow{q}_2$. Therefore, the degree of extension of the phase element at time t_n is characterized by the eigenvalues of the matrix K_n

$$\mathrm{K}_n = \mathrm{P}_n\mathrm{P}_{n-1}...\mathrm{P}_2\mathrm{P}_1\mathrm{P}_1^T\mathrm{P}_2^T...\mathrm{P}_{n-1}^T\mathrm{P}_n^T. \tag{4}$$

Thus, the speed of stretching or compression of the initially selected phase volume in time is determined by the following relationship:

$$\eta_{ni} = \frac{1}{n}\ln\left(\frac{\lambda_{ni}}{\lambda_{ni+1}}\right), \tag{5}$$

where λ_{ni} (i = 1,2...) are the eigenvalues of matrix K_n. To determine the statistical characteristics of the process, we need to repeat the calculation of η_{ni} many times

and average over the number of realizations m, therefore, we can calculate η_{ni} and its dispersion $\sigma_n(\eta, m)$

$$\sigma_n(\eta, m) = \left[m^{-1} \sum_{i=1}^{m} \eta_{ni}^2 - \left(m^{-1} \sum_{i=1}^{m} \eta_{ni} \right)^2 \right]^{1/2}. \tag{6}$$

It should be noted that in the process of studying the behaviour of the test phase volume we use various matrices A and B which satisfy the conditions

$$\mathrm{A}\overrightarrow{q} \neq \mathrm{B}\overrightarrow{q} \neq \overrightarrow{q}, \qquad \mathrm{AB} \neq \mathrm{BA}, \qquad |\det \mathrm{A}| = |\det \mathrm{B}| = 1. \tag{7}$$

One or the other matrix is chosen randomly using depending on the parameter p from the interval (0;1) using the standard program of the random number generation. In (7) the meaning of the first two expressions is clear and the last requirement follows from the Liouville's theorem, which ensures the conservation of the phase volume in time. In the calculations of the evolution of the selected phase volume we used the following matrices:

$$A(k_i) = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_1^{-1} & 0 \\ 0 & 0 & 0 & k_2^{-1} \end{pmatrix}, \quad B(h_i) = \begin{pmatrix} 2h_1 & 0 & 0 & h_1 \\ 0 & h_1^{-1} & 0 & 0 \\ 0 & 0 & h_2 & 0 \\ h_2^{-1} & 0 & 0 & h_2^{-1} \end{pmatrix},$$

$$C(\alpha) = \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix}.$$

Here, k_i and h_i are random numbers with the normal distribution in the interval $[0, 1]$ and α is the angle with a random distribution in the interval $[0, 2\pi]$.

In order to calculate the statistical stretching of the selected phase volume, a corresponding computer code was written. Semi-axes values of the four-axis ellipsoid and the dispersion of the elongation of the test phase volume were calculated. Since we are dealing with a large number of random processes, we were interested in checking the central limit theorem of probability theory (Tutubalin 1992).

All calculations are done in two modes: in the first case (Fig. 1a), A and B are taken as random matrices, which are given by the value of the parameter p. If $p \leq 0.5$ then $P_i = A$ and in the opposite cases $P_i = B$; in the second case (Fig. 1b), three random matrices A, B, C are used, where $P_i = A$ when $0 \leq p \leq 0.33$. If the value of probability p is in range $0.33 \leq p \leq 0.66$ then $P_i = B$, in other cases $P_i = C$. According to our estimations, the number of realization in all calculations was taken to be $m = 500$, because more than this value, its contribution to the accuracy of calculations is almost not noticeable. Thus, after each "push", the eigenvalue of the matrix is calculated and the stretching velocity of the test phase volume is determined from them, then its elongation dispersion is calculated until its value is less than ϵ, where $\epsilon = |\sigma_n(\eta, m) - \sigma_{n-1}(\eta, m)| = 10^{-6}$.

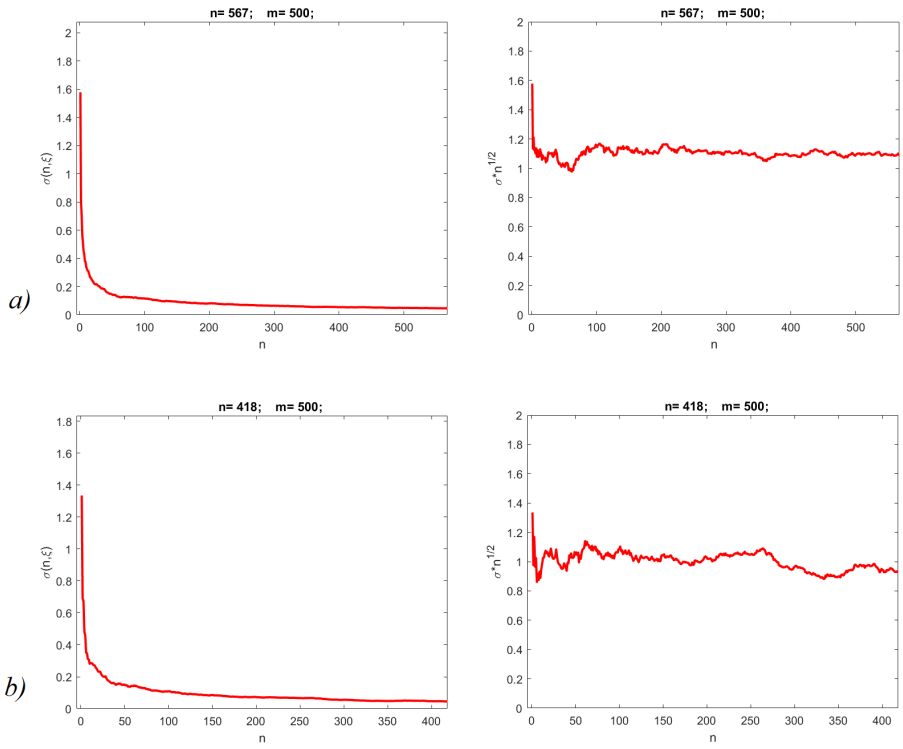


Figure 1 Dependencies of σ and $\sigma\sqrt{n}$ on step n. Calculations were done with two (a) and three (b) types of random matrices.

3 Application of the method to a three-dimensional case

The consideration of the 3-dimensional case in ordinary physical space is of special interest. This case is not connected with the process of phase mixing, but in cosmogony of some astrophysical objects, in principle, can be found a non-stationary state where stochasticity or violent turbulence takes place. It is also known that at an earlier stage of evolution of the Universe there was a period of some strong turbulence (see, e.g., Zel'dovich and Novikov 1983).

Using the model of chaotic impacts via method of random matrices developed by us, we have studied the behaviour of initial element over time for some specific cases, without specification of the exact cosmogonic application. The point is that there are many variants for selecting these matrices, and the corresponding analysis of the statistical effect is not an easy work. Therefore, below we are giving the results of the calculation for the simplest case where we have implemented the numerical calculations for the following random matrices:

$$A\left(k_i\right) = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & \left(k_1 k_2\right)^{-1} \end{pmatrix}, \qquad B\left(\alpha\right) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix},$$

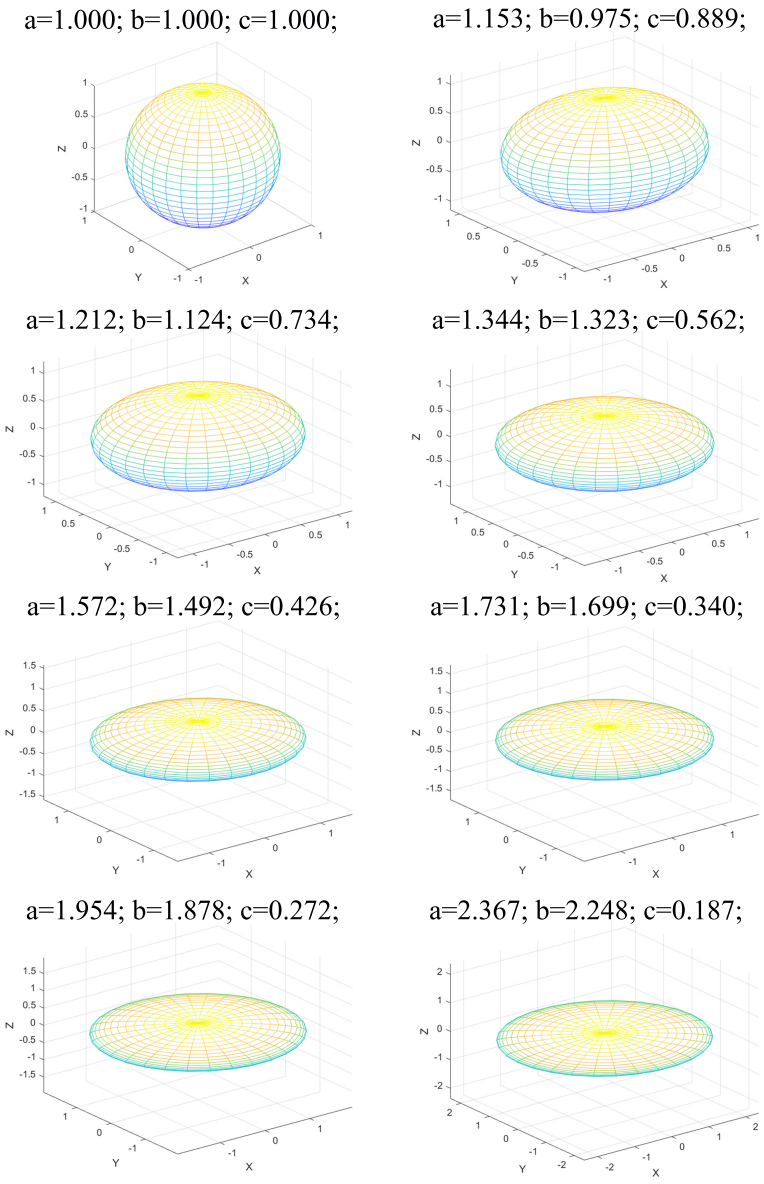


Figure 2 Behaviour of spherical volume in the 3D space after a series of random impacts.

which also satisfy the conditions in (7). The results of our calculations show that after the series of random impacts the initial spherical volume tries to form a plate nearly disk-like shape (Fig. 2).

It is certainly an unexpected result. That is why we also tested other selected cases of random matrices A and B. Note that the striving tendency to a disk-like shape is not observed in all cases. On the basis of the above now we have a new task to analyze the more complex types of random matrices and improve the model by

increasing the number of these matrices. This is the only way to determine the real statistical effect of these random impacts on the given spherical volume in the 3D case.

4 Conclusions

It should be recalled again that the physical process discussed in this work in the form of compulsive phase mixing occurs only at a clearly low value of the virial ratio and due to the strong variability of the gravitational potential both in time and in coordinate space. Due to these properties, this type of mixing is different from other types of stochastic processes (see Krylov 1980, Rastorguev and Sementsov 2006, and references therein).

In the framework of our work, the following results are obtained:

• a technique is proposed for analyzing the statistical effect of random impacts on a selected volume element in the case of four-dimensional phase space;

• characteristic time decreases noticeably by increasing the number of random matrices;

• in the four-dimensional phase space the final form of the test volume element is a filamentous structure elongated in one direction;

• the application of the method for studying random impacts on a volume element in the case of three-dimensional space is carried out;

• we verified the fulfillment of the central limit theorem of probability theory, which obeys the law $\sigma_n(\eta, m) \sim n^{-1/2}$.

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