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Periodic solutions of the restricted three body problem for small μ and the motion of small bodies of the Solar System

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The plane circular restricted three body problem is considered for small mass ratios μ of the main bodies. All the limit problems as $\mu \to 0$: the two-body problem, Hill's problem, the intermediate Hénon problem and the basic limit problem, are found using Power Geometry. In each of these problems, we isolate solutions which are the limits of the periodic solutions of the restricted problem as $\mu \to 0$, as well as the limits of the families of periodic solutions (which are called generating families). Using the generating families, we study the families for small $\mu > 0$ which begin with the retrograde (family h) and direct (family i) circular orbits of infinitesimal radius around the body of greater mass. It is shown that, as μ increases, there is a small change in the structure of the family h but family i undergoes infinitely many self-bifurcations with the formation of an infinite number of closed subfamilies, each of which only exists in a certain range of values of μ. We also give a theory of the formation of horseshoe-shaped orbits and orbits in the form of "tadpoles", and the structure of the basic families containing periodic solution with these orbits is indicated. The implications of these results for the motion of asteroids, satellites, planetary rings, and particles of the Kuiper belt are discussed.

Keywords: Restricted 3-body problem; Periodic solutions; Solar System

1 Introduction

Suppose three point bodies P_1, P_2 and P_3 move in a single plane under the action of Newton's law of gravitation. The bodies P_1 and P_2 have masses m and m_2 respectively while the mass of the body P_3 is so small that its effect on the bodies P_1 and P_2 can be neglected. We will say that the mass of the body P_3 is equal to zero. Then, the body P_2 executes a Kepler motion with respect to the body P_1. If the body P_2 moves along a circle, the problem of the motion of the body P_3 is referred to as a circular restricted three-body problem or, briefly, as the restricted problem.

We assume that the units of mass, time and distance are chosen such that the sum $m + m_2$, the gravitational constant, the distance P_1P_2, and the angular velocity

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of the body P_2 with respect to the body P_1 are equal to unity. The single parameter will then be $\mu = m_2 \in [0, 1/2]$. If a system of coordinates rotating together with the body P_2 is introduced, then, in this (synodic) system of coordinates with its center at P_1, the position x_1, x_2 of the body P_3 is described by a Hamiltonian system with two degrees of freedom and a single parameter μ (Bruno 1990; 1994, Ch. III, Sect. 1)

$$\dot{x}_j = \partial H/\partial y_j, \qquad \dot{y}_j = -\partial H/\partial x_j, \quad j = 1, 2, \tag{1}$$

where

$$H = H_0 + \mu R, \quad H_0 = (y_1^2 + y_2^2)/2 + x_2 y_1 - x_1 y_2 - r^{-1}, \tag{2}$$

$$R = r^{-1} + x_1 - r_2^{-1}, \quad r = \sqrt{x_1^2 + x_2^2}, \quad r_2 = \sqrt{(x_1 - 1)^2 + x_2^2}.$$

A derivative with respect to time t is denoted by dot.

When $\mu \neq 0$, the problem is not integrable in a finite form. The families of periodic solutions are of the greatest interest as they form a kind of a skeleton of a certain part of the phase space. For a fixed value of the parameter $\mu \neq 0$, the periodic solutions of the Hamiltonian system (1) form one-parameter families and, in the case of a variable μ, two-parameter families (Bruno 2006).

The aim of this paper is to give a survey of the results on the families of periodic solutions of the restricted problem for small μ. The interest to these results is motivated by their application to the study of motion of small bodies in the Solar system (particles of rings planetary belts of giant planets, asteroids, satellites, particles of the Kuiper belt) and space probes.

The majority of the publications are concerned with computation of the families of periodic solutions for fixed values of μ, where no developed theory is required. However, there is another approach: to consider limit problems as $\mu \to 0$ and their regular and singular perturbations for $\mu > 0$. This approach makes it possible to construct a theory of the families of periodic solutions for small μ (Bruno, Varin 2007; 2008).

We will consider the restricted problem for small $\mu > 0$ as a perturbation of this problem with $\mu = 0$. Then the periodic solutions present for $\mu = 0$ undergo either regular, or singular perturbations. All periodic solutions with regular perturbations are described in the book (Bruno 1990; 1994). The singular perturbations, however, originate from the three sources: (a) the body $P_2 = \{x_1 = 1, x_2 = 0\}$; (b) the arc-solutions twice intersecting the body P_2; (c) the circle of stationary points $x_1^2 + x_2^2 = 1$, $y_1 = -x_2$, $y_2 = x_1$. These singular perturbations are the main concern of this paper (see in more detail in (Bruno, Varin 2007)).

An *orbit* is the projection of the solution $x_j(t), y_j(t)$ $(j = 1, 2)$ of the system (1) on the plane x_1, x_2. System (1) is invariant under the substitution

$$t, x_1, x_2, y_1, y_2 \longrightarrow -t, x_1, -x_2, -y_1, y_2, \tag{3}$$

which is the *symmetry* of the system. The plane $x_2 = y_1 = 0$ is invariant under the symmetry (3); it is called the *symmetry plane* Π (Bruno 1990; 1994, Chap. III). The solutions of (1) invariant under the substitution (3) are *symmetric*. Symmetric periodic solutions intersect the symmetry plane twice. These intersections are convenient for tracking mutual dispositions of these solutions. The curve of the intersection of a family with the symmetry plane is called the *characteristic* of the family. The families

of symmetric periodic solutions to a Hamiltonian system with two degrees of freedom are studied in (Bruno 2006, § 5). Every symmetric periodic solution M of a family F has a period T, a trace Tr of the monodromy matrix of the variational system, and two points of intersection with the symmetry plane Π (after the half-period). The value of the Hamiltonian function or a coordinate of the intersection with the symmetry plane can be chosen as a parameter along the family.

2 The neighborhood of the body P_2

In Bruno (1998, 2000, Chap. IV, Section 4) and in Bruno & Varin (2007), the limit problems for the system (1), (2) near the body P_2 for small $\mu \to 0$ were derived with the help of Power Geometry. It was established that the system (1) has 4 limit problems: (a) the two body problem for P_2 and P_3; (b) Hill's problem; (c) the intermediate Hénon problem; and (d) the basic limit problem derived from the system (1), (2) for $\mu = 0$. Among these four problems, only Hill's problem is not integrable.

The two body problem for P_2 and P_3 has two families of periodic solutions with circular orbits: the family f with retrograde direction of the motion, and family g with direct motion. Only these two families are preserved under the perturbations of the restricted problem, while other families are destroyed (Bruno 1990, 1994, Introduction, p. 8; Hénon 1997, § 5.6). In addition, we note the existence of hyperbolic fly-by orbits of the body P_3 close to the body P_2.

Hill's problem (Hénon 1969a, b) has two stationary points L_1, L_2. The most detailed description of its families was given by Hénon (1969a, b), who plotted their characteristics in the symmetry plane Π.

The intermediate problem was introduced and studied by Hénon (1969). Its periodic solutions were used as generating ones in the search for periodic quasi-satellite orbits of the restricted problem (see references in Bruno & Varin (2007)).

3 The basic limit problem

The basic limit problem is the two body problem for P_1 and P_3 in which the part of the phase space corresponding to the body $P_2 = \{x_1 = 1, x_2 = 0\}$ is removed. The two body problem for P_1 and P_3 has the following families of symmetric periodic solutions:

Id – direct with circular orbits,

Ir – retrograde with circular orbits,

$E_N^{\pm}$ – with sidereal (i.e. fixed coordinate system) elliptic orbits with a fixed major semiaxis a and an arbitrary eccentricity $e \in (0, 1]$. The mean motion n is fixed on the family; $\operatorname{sign} n$ indicates the direction of the motion (the direct +, and retrograde –), and $N = |n| = a^{-3/2}$.

Let the set $\mathcal{D}$ consist of all those and only those trajectories sidereal orbits of which are ellipses. We consider intersections of trajectories of the set $\mathcal{D}$ with the symmetry plane $\Pi = \{x_2 = y_1 = 0\}$. It is convenient to introduce coordinates $\tilde{a}$, $\tilde{e}$ in this plane (Bruno 1990, 1994, Ch. III):

$$\tilde{e} = x_1 y_2 |y_2| m^{-1}, \quad \tilde{a} = x_1 (2 - |\tilde{e}|)^{-1}.$$

Then

$$a = |\tilde{a}|, \quad e = |1 - |\tilde{e}||, \quad n = \operatorname{sign} \tilde{e} a^{-3/2}.$$

In coordinates $\tilde{a}$, $\tilde{e}$, the intersection of the set $\mathcal{D}$ with the plane Π is the band

$$\tilde{a} \in \mathbb{R} \backslash 0, \quad \tilde{e} \in [-2, 2].$$

Two straight lines $\tilde{e} = 1$ and $\tilde{e} = -1$ correspond to circular orbits with the period T and the trace Tr:

$$T = 2\pi |n - 1|^{-1}, \quad \mathrm{Tr} = 2 \cos T$$

if $n \neq 1$. If $n = 1$, then the corresponding points $\tilde{a} = \pm 1$, $\tilde{e} = 1$ are stationary. On the straight line $\tilde{e} = 1$, sidereal motion is direct (it is the family Id), and on the straight line $\tilde{e} = -1$, the motion is retrograde (it is the family Ir).

Four bands $\tilde{e} \in (-2, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$ correspond to elliptic sidereal orbits with direct motion if $\tilde{e} > 0$, and to retrograde motion if $\tilde{e} < 0$. For a fixed $a = |\tilde{a}|$ with a rational $N = a^{-3/2} = (p + q)/p$, the corresponding synodic orbits are periodic with the period $T = 2\pi p$ and the trace $\mathrm{Tr} = 2$ (it is the family $E_N^{\pm}$). All the points of the vertical segments $\tilde{a} = \pm N^{-2/3} = \mathrm{const}$ correspond to a unique family $E_N^{\pm}$ (Bruno 1990, Fig. 11; 1994, Fig. III.9).

In the basic limit problem, the parts of solutions of the two body problem for P_1 and P_3 which begin and end by points of collision of the body P_3 with the body P_2 are of great interest. They are called *arc-solutions* or *solutions with consecutive collisions*. In the restricted problem, the arc-solutions play almost the same part as periodic solutions.

Every arc-solution belongs to a certain solution of the two body problem. Symmetric arc-solutions constitute a denumerable set S of one-parameter families A_i, B_j, C_{kl}, where integers $i \geq 0$, $j \geq 1$, $l \geq k \geq 1$. For the first time, the families S were found by Hénon (1968); then their theory was developed in (Bruno 1990, 1994, Chap. III-VI; Hénon 1997). We remark that each symmetric arc-solution intersects with the plane Π at a unique point, but their one-parameter family intersects Π by a curve, which is called *characteristic* of the family. The orbits of arc-solutions of the families A_i, B_j, C_{kl} were given in (Bruno 1990, 1994, Chap. IV). Figure 1a shows characteristics of the families A_i, C_{kl}, and Id in coordinates $\tilde{a}$, $\tilde{e}$ in the left upper quarter of the plane Π ($\tilde{a} < 0$, $\tilde{e} \geq 0$).

4 Generating and generated families of periodic solutions

Suppose a periodic solution M_μ of the restricted problem (1), (2), which exists for a certain $\mu > 0$, can be continued continuously up to $\mu = 0$ and, in the limit, gives a solution M_0 (not a point) of one of the limit problems. This limit is called a generating periodic solution. According to Hénon (1997, § 2.10), there are three species of generating periodic solutions depending on the limit M_0 in the main limit problem:

The first species. All points of the solution M_0 are separated from the body P_2.

The second species. The solution M_0 has at least one point on the body P_2 and one point outside the body P_2.

The third species. The solution M_0 lies as a whole in the body P_2.

A generating solution of the first species is a periodic solution of the synodic problem for the two bodies P_1 and P_3. A generating solution of the second species consists of several arc-solutions of the basic limit problem. Since the restricted problem has the integral H from (2), all the arc-solutions which are parts of generating periodic solution have the same value of the integral H or the Jacobi constant $C = -2H$. A generating periodic solution of the third species is a periodic solution of Hill's problem or the intermediate Hénon problem. The limit of the family of periodic solutions as $\mu \to 0$ is called the *generating family.* It can consist of generating periodic solutions of different species.

Various parts of different families of generating periodic solutions Id, Ir, $E_N^{\pm}$, A_i, B_j, C_{kl} are united into one generating family according to bifurcations taking place at $\mu = 0$. These bifurcations were described for the most part in the books by Hénon (1997, 2001) (see also (Bruno 1990; 1994; Bruno, Varin 2007; 2008)).

We remark that the parts of generating families Id, Ir, $E_N^{\pm}$, A_i, B_j, C_{kl} are relatively easy to compute. On the contrary, computation of the generated families for $\mu > 0$ presents sometimes an insurmountable problem. In the book (Hénon 1997, Ch. 10), the principal generating families were described. A detailed comparison of

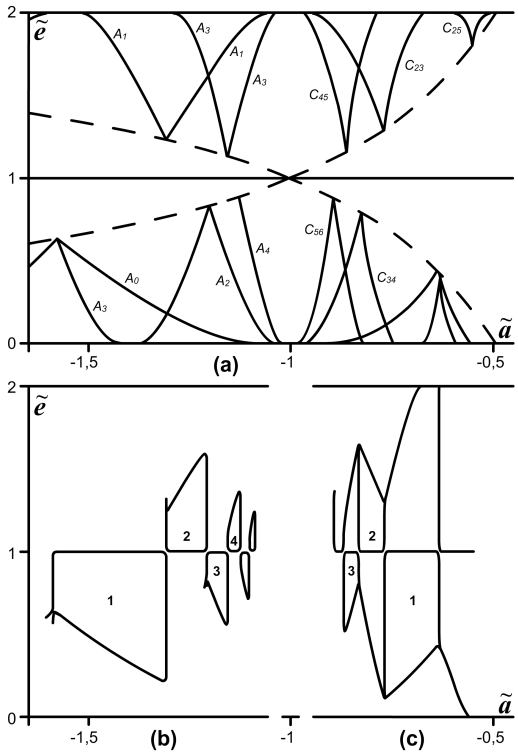


Figure 1 Parts of the symmetry plane Π with: parts of characteristics of families A_j, $C_{k,l}$, and Id for $\mu = 0$ (a); parts of characteristics of families of periodic solutions close to $\tilde{e} = 1$ for $\mu = 5.178 \times 10^{-5}$ (b); parts of characteristics of the family i for $\mu = 5 \times 10^{-5}$ (c).

the generating families with the generated ones was performed for the families h, c, i.

The family h begins with the retrograde circular orbits of infinitely small radius around the body P_1 of bigger mass. The generating family h is described in Bruno & Varin (2006; 2009a); the generated family for $\mu = \mu_J \approx 10^{-3}$ corresponding to Sun (P_1) – Jupiter (P_2) case in Bruno & Varin (2006; 2009a); for $\mu = \mu_M \approx 10^{-2}$ corresponding to Earth (P_1) – Moon (P_2) case in Brouke (1968); for $\mu = 0.1, 0.2, 0.3, 0.4, 0.5$ in Bruno & Varin (2007; 2009a,c). The generated family is very close to the generating one for $\mu \le 0.1$.

The family c begins at the collinear stationary point L_1. For $\mu = 0$, the generating family c consists of two parts: the family c of the Hill problem and the part with $a < 1$ of the family B_1 of arc-solutions. This family was described in detail in (Bruno, Varin 2008). For $\mu = 5 \times 10^{-5}$, the family c was computed in (Bruno, Varin 2009b); it is very close to the generating family.

The family i begins with the direct circular orbits of infinitely small radius around the body P_1 of bigger mass. For $\mu = 0$, the generating family has a very complicated structure. It includes parts of the families Id, $E_N^{\pm}$, B_1, and $C_{k,k+1}$. It is described in detail in (Bruno, Varin 2007; 2008). For $\mu = 5 \times 10^{-5}$, the family i was computed in (Bruno, Varin 2009b); its initial part is very close to the generating family. For this μ, the characteristic in the left upper quarter of the symmetry plane Π ($\tilde{a} < 0$, $\tilde{e} \ge 0$) is shown in Fig. 1c in coordinates $\tilde{a}, \tilde{e}$: its horizontal parts correspond to the parts of the family Id; vertical parts correspond to the parts of the families $E_N^{\pm}$; curvilinear parts correspond to the parts of the families $C_{k,k+1}$. Above and below the horizontal line $\tilde{e} = 1$, the characteristic forms certain structures, which we call *bubbles* (they are marked by numbers $k = 1, 2, 3$). It turns out that these bubbles bifurcate into closed characteristics at a certain value of μ and form independent families i_k. The families i_k exist only in the intervals $\mu \in [\mu'_k, \mu''_k]$. For $k = 1, 2$, this process is shown in Figs. 2 and 3 respectively. At $\mu = \mu''_k$, the family i_k collapses into a single periodic solution (Bruno, Varin 2007; 2009d).

Similar bubbles can be observed close to the family Id for $a > 1$. Fig. 1b (which was sent to the first author by G. Vojatzis in 2006) shows characteristics for $\mu = 5.178 \times 10^{-5}$. The horizontal parts of these characteristics correspond to the family Id; the vertical parts correspond to families $E_N^{\pm}$; and curvilinear parts correspond to families A_i. The bubbles above and below the horizontal line $\tilde{e} = 1$ (marked by the numbers $k = 1, 2, 3, 4$) also close upon themselves and form closed families (Vojatzis, Kotoulas 2005). We remark that Fig. 1b and 1c are almost symmetrical with respect to the vertical line $\tilde{a} = -1$.

5 Horseshoes and tadpoles

The families of periodic orbits in the form of horseshoes and tadpoles are present for $\mu > 0$ close to the circle of stationary points

$$x_1^2 + x_2^2 = 1, \quad y_1 = -x_2,\ y_2 = x_1,$$

which is present at $\mu = 0$. The point $x_1 = 1$, $x_2 = 0$, where the body P_2 is placed, is excluded from these stationary points. Only three Lagrange points are generating:

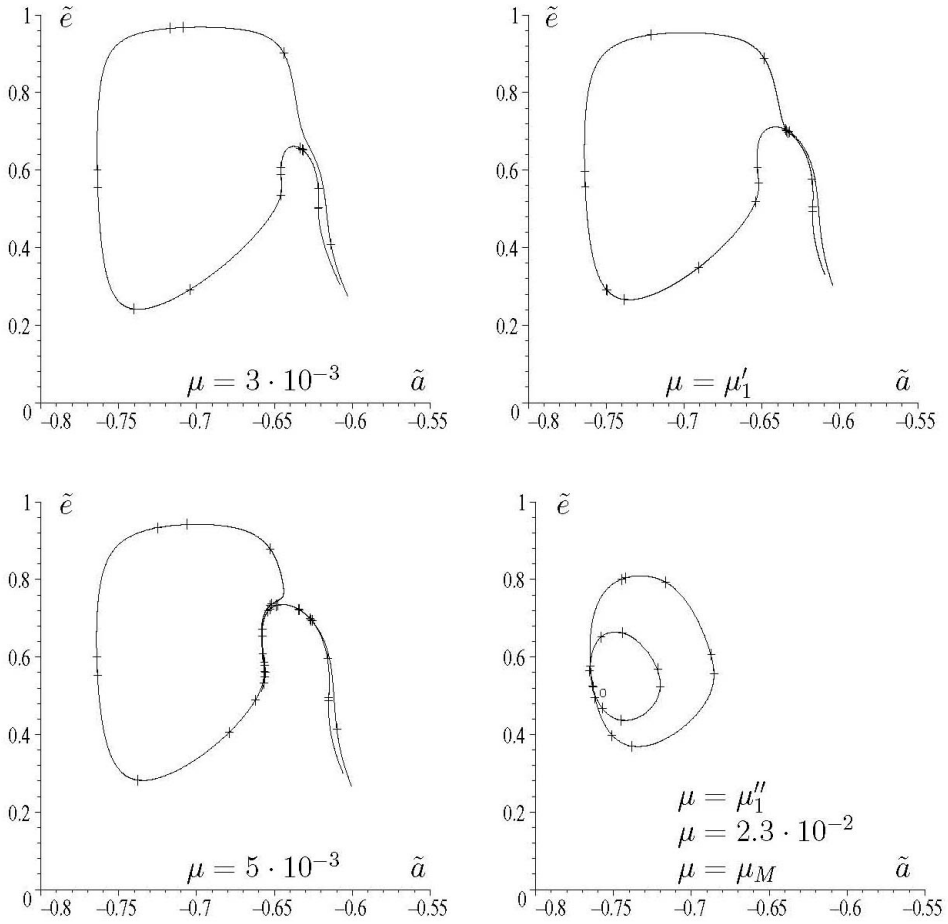


Figure 2 Formation, evolution, and collapse of the closed family i_1 in its characteristics in the left half-plane Π.

$L_3 = \{x_1 = -1, x_2 = 0\}$, $L_{4,5} = \{x_1 = 1/2, x_2 = \pm\sqrt{3}/2\}$. For small μ, there exist families of periodic solutions the orbits of which either encircle all three points and have the form of horseshoes, or encircle one of the points L_4, L_5 and have the form of tadpoles. The theory of these families is given in Bruno & Varin (2007, Section 7).

6 Applications

The family i for $\mu = \mu_J = 0.00098$ (Sun – Jupiter case) is related to the motion of asteroids (Bruno, Varin, 2011) for $\mu = 6.7 \times 10^{-6}$ (Saturn – Mimas) is related to the motion of particles in the Saturn ring (Franklin, Colombo 1970). Similar families with $a > 1$ for $\mu = 5.178 \times 10^{-5}$ are related to the motion of particles in the Kuiper belt (Kotoulas, Vojatzis 2004), (Vojatzis, Kotoulas 2005a, b),

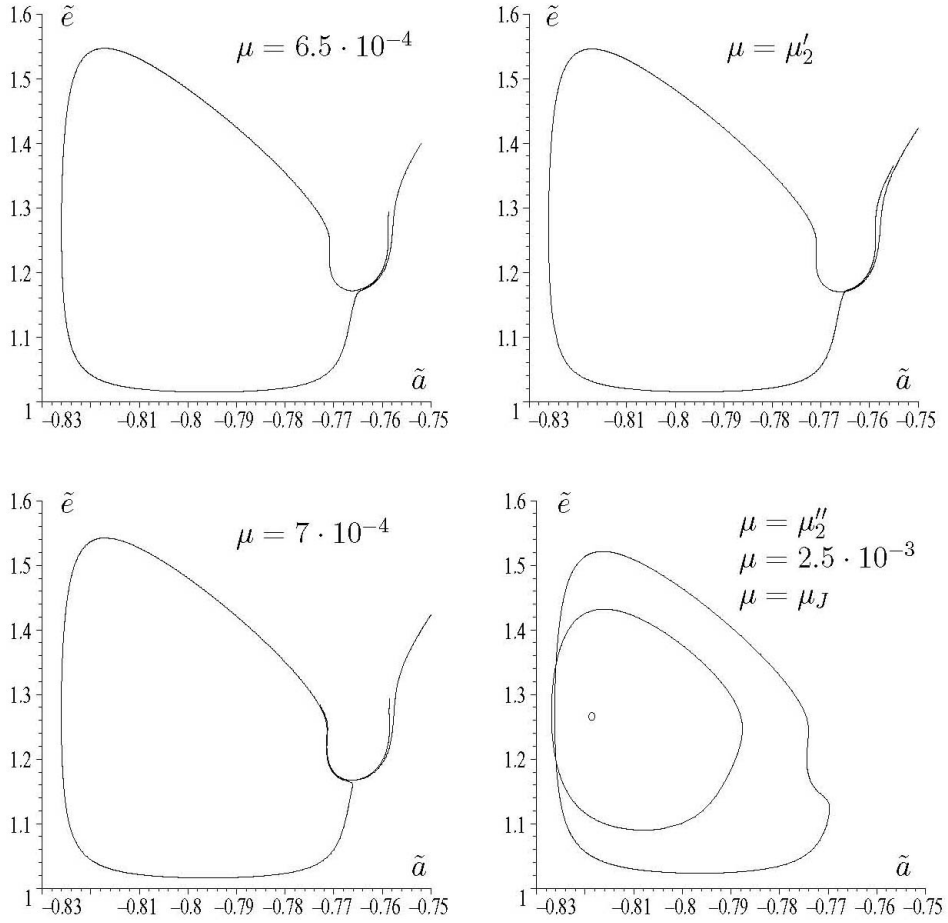


Figure 3 Counterpart of Fig. 2 for the closed family i_2.

The family c with horseshoe orbits for $\mu = 3.5 \times 10^{-9}$ (Saturn – Janus) is related to the motion of a satellite of Saturn called Epimetheus (1980S3) (Bruno, Varin 2007). In addition, for various small μ, various families of periodic solutions correspond to periodic motions of space probes.

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