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A STUDY OF THIN DISK WITH RADIAL FLOW FOR PERFECT FLUID DISK IN KERR GEOMETRY

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Abstract:

In this paper we report a study of thin disk formulation with radial flow of perfect fluid disk rotating around black hole in linearised Kerr geometry. The effects of rotation through an angular momentum **a** is obtained on pressure, velocity and density for co-rotating and counter – rotating disk

Key words: Black hole, thin disk.

1. INTRODUCTION:

One of the most important aspects in the study of rotating black hole is the formation of thin disk. We have developed fully general relativistic approach for the dynamics of thin accretion disk with an angular momentum **'*a*'** for non self gravitating perfect fluid in the linearised Kerr back ground geometry.

We have considered the case when the pressure force is negligible, so that the disk is essentially a thin disk, confined to the equatorial plane. We include accretion by assuming a non zero velocity in the radial direction. The effect of rotation of black hole clearly seen by comparing these solutions with their Schwarzchild counterpart.

2. MATHEMATICAL FORMULATION:

Space time metric of rotating black hole of mass M and angular velocity a (in unit) ***c*** is given by Kerr metric (1963) which in Boyer Lindquist (t, r, θ, ϕ) co-ordinate may be written as Misner, Thorne and Wheeler 1972.

$$ds^2 = -\left(1-\frac{2mr}{\Sigma}\right)c^2dt^2 - \frac{4mra}{\Sigma}\sin^2\theta\, dt\, d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 + \left(\frac{\Delta}{\Sigma}\right)\sin^2 d\phi^2$$

where $\sum = r^2 + a^2 \cos^2\theta, \qquad \Delta = \left(r^2 + a^2 - 2mr\right)$

$$A = \left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta$$

$m = \dfrac{MG}{c^2}$, c being the velocity of light.

The non-zero metric component $g_{ij}\left(i, j = 0,1,2,3\right)$, of the above as follows

$$\left[\begin{array}{ll} g_{t\phi} = -\dfrac{2mra\sin^2\theta}{\sum}, & g_{\phi\phi} = \dfrac{A}{\sum}\sin^2\theta \\ g_{tt} = -\left(1 - \dfrac{2mr}{\sum}\right), & g_{rr} = \dfrac{\sum}{\Delta}, g_{\theta\theta} = \sum \end{array}\right] \qquad ----(2.2)$$

The equation governing the motion of perfect fluid is described by law of conservation of energy momentum tensor.

$$T^{ij}_{,j} = 0 \qquad ----(2.3)$$

where $T^{ij} = -pg^{ij} + \left(p + \rho c^2\right)u^i u^j$

where p is the pressure, ρc^2 is the energy density including rest mass energy and u^i is the four velocity satisfying the normalization relationship

$$u_i u^i = 1$$

The equation (2.1) are expressed in terms of equation of continuity and the momentum equation as below:

Continuity equation

$$\rho, i\, u^i + \left(\rho + \frac{p}{c^2}\right)u^i, i = 0 \qquad ----(2.4)$$

and momentum equation

$$\left(\rho + \frac{p}{c^2}\right)u^i_{ij}; u^j = \frac{1}{c^2}\left(g^{ij} - u^i u^j\right)p_{ij} \qquad ----(2.5)$$

The connection co-efficients Γ^i_{jk} are taken from (Mishra and Chakraborty, 2003).

$$\Gamma^i_{jk} = g^{kh}[i, j\ h] \qquad ----(2.6)$$

The continuity equation as obtained from (2.4) is

$$\left(\rho+\frac{p}{c^2}\right)\left[\frac{\partial v^\alpha}{\partial x^\alpha}-\Gamma^0_{0\alpha}v^\alpha+c\Gamma^\alpha_{0\alpha}+\Gamma^\beta_{\alpha\beta}v^\alpha-\Gamma^0_{\alpha\beta}v^\alpha\frac{v^\beta}{c}\right]+\left(\frac{\partial\rho}{\partial t}+v^\alpha\frac{\partial\rho}{\partial x^\alpha}\right)-\frac{1}{c^2}\left(\frac{\partial\rho}{\partial t}+v^\alpha\frac{\partial\rho}{\partial x^\alpha}\right)$$

$$-\frac{1}{cu^{02}}\left(g^{00}\frac{\partial p}{c\partial t}+g^{0\alpha}\frac{\partial p}{\partial x^\alpha}\right)=0 \qquad ----(2.7)$$

The momentum equation, as obtained from (2.5) is

$$\left(\rho+\frac{p}{c^2}\right)u^{02}\left[\frac{\partial v^\alpha}{\partial t}+v^\beta\frac{\partial v^\alpha}{\partial x^\beta}+c^2\left(\Gamma^\alpha_{00}-\frac{v^\alpha}{c}\Gamma^0_{00}\right)+2cv^\beta\left(\Gamma^\alpha_{0\beta}-\frac{v^\alpha}{c}\Gamma^\alpha_{0\beta}\right)+v^\beta v^r\left(\Gamma^\alpha_{\beta r}-\frac{v^\alpha}{c}\Gamma^0_{\beta r}\right)\right]$$

$$=-\left(g^{\alpha 0}-\frac{v^\alpha}{c}g^{00}\right)\frac{\partial p}{c\partial t}-\left(g^{\alpha\beta}-g^{0\beta}\frac{v^\alpha}{c}\right)\frac{\partial p}{\partial x^\beta} \qquad ----(2.8)$$

3. DYNAMICS OF DISK WITH RADIAL FLOW

The general set of equations governing the dynamics of disk with radial in locally non-rotating frame (LNRF) will be more transparent because it cancels out as much as possible the effect of frame dragging of Block hole's rotation Bardeen (1972, 1973), so we may express equation of motion in term of 3 velocity in L.N.R.F which is written below:

$$\frac{v^\alpha}{c}=\frac{c\,\lambda^\alpha_{(0)}+\lambda^\alpha_{(\beta)}v^\beta}{\left(\lambda^\alpha_{(0)}+\lambda^0_{(\beta)}v^{(\beta)}\right)}\ .\text{The tetrad} \qquad ----(3.1)$$

$\lambda^{(i)}_j$ and $\lambda^i_{(j)}$ appropriate for the transformation in Kerr geometry, which in terms of the metric component in Boyer-Lindquist co-ordinate system may be written as:

$$\lambda^{(i)}_j=\begin{vmatrix}\left(\frac{\Delta\Sigma}{\Delta}\right)^{1/2} & 0 & 0 & 0\\ 0 & \left(\frac{\Sigma}{\Delta}\right)^{\frac{1}{2}} & 0 & 0\\ 0 & 0 & \Sigma^{1/2} & 0\\ \left(\frac{-A}{\Sigma}\right)^{1/2}\frac{2gmar\sin\theta}{A} & 0 & 0 & \left(\frac{A}{\Sigma}\right)^{1/2}\sin\theta\end{vmatrix}$$

$$\lambda^{i}_{(j)} = \begin{vmatrix} \left(\frac{A}{\Delta\Sigma}\right)^{1/2} & 0 & 0 & 0 \\ 0 & \left(\frac{A}{\Sigma}\right)^{1/2} & 0 & 0 \\ \left(\frac{A}{\Sigma\Delta}\right)^{1/2}\frac{2mar}{A} & 0 & 0 & \left(\frac{\Sigma}{A}\right)^{1/2}\cos ec\theta \end{vmatrix}$$

$$----(3.2)$$

we limit ourselves to the case $v_0^{(r)}$, $v_0^{(\phi)}$, $p, \rho, n \neq 0$ and are function of r *and* θ only, and only $v^{(\theta)} = 0$. Thus the equations reported by (Mishra and Chakraborty, 2003) governing the radial flow reduces to stationary state with axi symmetric as follows

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\left[\left(1-\frac{2m}{r}\right)v_0^{(r)}\frac{\partial v_0^{(r)}}{\partial r} - \frac{1}{r}\left(\frac{1-2m}{r}\right)v_0^{(\phi)^2} + \frac{mc^2}{r^2}\left(\frac{1-v_0^{(r)^2}}{c^2}\right) - \frac{6amc}{r^3}\left(\frac{1-2m}{r}\right)^{1/2}\sin\theta v_0^{(\phi)} \times\left(\frac{1-v^{(r)^2}}{c^2}\right)\right] = -\left(\frac{1-v^2}{c^2}\right)\left(\frac{1-2m}{r}\right)\frac{\partial p}{\partial r} \qquad ----(3.3)$$

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\left[\left(-\frac{\cot\theta}{r}\right)\left(\frac{1-2m}{r}\right)^{1/2}v^{(\phi)^2}\right] = -\left(1-\frac{v^2}{c^2}\right)\frac{1}{r}\left(\frac{1-2m}{r}\right)^{-1/2}\frac{\partial p}{\partial\theta} \quad ----(3.4)$$

$$\frac{\partial}{\partial r}\log v^{(\phi)} + \frac{1}{r}\left(1-\frac{2m}{r}\right)^{-1}\left(\frac{1-3m}{r}\right) + \frac{6amc}{c^2r^3}\sin\theta\left(1-\frac{2m}{r}\right)^{-1/2}v^{\phi} = 0 \quad ----(3.5)$$

$$\left(\rho_0 + \frac{p_0}{c^2}\right)\left[\left(1-\frac{2m}{r}\right)\left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(r^2v^{(r)}\right) + \frac{6am}{cr^3}\sin\theta v^{(r)}v^{(\phi)}\right] + \left(1-\frac{2m}{r}\right)v^{(r)}\frac{\partial}{\partial r}\left(\rho + \frac{p}{c^2}\right) = 0$$

$$n\left[\left(1-\frac{2m}{r}\right)\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2v^{(r)}\right) + \frac{6am}{cr^3}\sin\theta v^{(r)}v^{(\phi)}\right] + \left(1-\frac{2m}{r}\right)v^{(r)}\frac{\partial n}{\partial r} \quad ----(3.6)$$

$$+\frac{n}{\rho+\frac{p}{c^2}}\times\left\{-\frac{1}{c^2}\left(1-\frac{2m}{r}\right)v^{(r)}\frac{\partial p}{\partial r}\right\}=0 \qquad ----(3.7)$$

and $$\left(1-\frac{2m}{r}\right)v^{(r)}\frac{\partial}{\partial r}\left(\frac{p}{nr}\right)=0 \qquad ----(3.8)$$

$r=\frac{cp}{cv}$ is ratio of specific heats

where n is number density of Baryons. The law of Thermodynamics which include the law of concentration of Baryon number.

$$\left(nu^i\right)_{ij}=0 \qquad ----(3.9)$$

In case of pressure less dust $p = 0$ and $\theta=\frac{\pi}{2}$. The fluid the confined to the equatorial plane. The Baryon conservation equation (3.7) together with continuity equation (3.6) can be written as

$$\frac{n\partial\rho}{\partial n}=\left(\rho+\frac{p}{c^2}\right)\frac{\partial n}{\partial r} \qquad ----(3.10)$$

while the adiabatic equation of state is written as

$$\frac{n\partial p}{\partial r}=rp\frac{\partial n}{\partial r} \qquad ----(3.11)$$

4. THIN DISK

We confine ourselves to the case of a cold disk with $p\approx 0$ contain at $\theta=\frac{\pi}{2}$ for negligible pressure the equation of stable yield $p\approx nMo$ under this limiting situation (3.10) and (3.11) become identically zero and we left with equation (3.3) (3.5) and (3.6) only which govern the steady state configuration

Rewriting these in dimensionless form, we have

$$\left(1-\frac{2}{R}\right)\frac{dv_0^{(\phi)}}{dR}+\left(1-\frac{3}{R}\right)\frac{v_0^{(\phi)}}{R}+\frac{6\alpha}{R^3}\left(1-\frac{2}{R}\right)^{\frac{1}{2}}v^{(\phi)^2}=0 \qquad ----(4.1)$$

$$\left(1-\frac{2}{R}\right)V_0^{(r)}\frac{dv^{(r)}}{dR}-\frac{1}{R}\left(1-\frac{2}{R}\right)V^{(\phi)^2}+\frac{1}{R^2}\left(1-V^{(r)^2}\right)+\frac{6\alpha}{R^3}\left(1-\frac{2}{R}\right)^{\frac{1}{2}}V^{(\phi)}\left(V^{(r)^2}-1\right)\text{———}(4.2)$$

and

$$\rho\left(1-\frac{2}{R}\right)\frac{1}{R^2}\frac{d}{dR}\left(R^2V^{(r)}\right)+\frac{6\alpha\rho}{R^3}\left(1-\frac{2}{R}\right)^{\frac{1}{2}}V^{(r)}\cdot V^{(\phi)}+\left(1-\frac{2}{R}\right)V^{(r)}\frac{\partial\rho}{\partial R}=0\text{———}(4.3)$$

where $\alpha=\frac{a}{m}$ and $R=\frac{r}{m}$. Solving (4.1) we obtained

$$v^{(\phi)}=\frac{A}{R}\left(1-\frac{2}{R}\right)^{\frac{1}{2}}\left(1+\frac{2\alpha A}{R^3}\right)\text{————}(4.4)$$

Substituting above $v^{(\phi)}$ in (4.2) we obtain an equation in $v_0^{(r)}$ and using it we obtain

$$V_0^{(r)^2}=1-\left(1-\frac{2}{R}\right)\left(1+\frac{A^2}{R^2}\right)\left(1+\frac{4\alpha A}{R^3}\right)\text{————}(4.5)$$

Using (4.4) in (4.3), we solve it to obtain

$$\rho=\frac{\rho_c}{R^2V_0^r}\left(1+\frac{2\alpha A}{R^3}\right)\text{————}(4.6)$$

In above A and ρ_c are two constants. In deriving (4.5) we have already used a boundary condition that

$$V_0^r=0 \text{ at } R=\infty$$

5. CONCLUSION

In the special case of Schwarzschild geometry we have

$$V_0^{(\phi)}=\frac{A}{R}\left(1-\frac{2}{R}\right)^{\frac{1}{2}}\text{————}(5.1)$$

$$V_0^{(r)^2} = \frac{2}{R} - \frac{A^2}{R^2}\left(1 - \frac{2}{R}\right) \qquad ----(5.2)$$

$$\rho = \frac{\rho_c}{R^2 V_0^{(r)}} \qquad ----(5.3)$$

Squaring and adding (5.1) and (5.2) we obtain $\frac{1}{2}\left(V_0^{(r)^2} + V_0^{(\phi)^2}\right) = \frac{1}{R}$ $----(5.4)$

In the Newtonian gravitational case, the potential energy per unit mass is $-\frac{1}{R}$ and therefore the total energy per unit mass which is the sum of Kinetic energy and potential energy becomes zero which is also the total energy at infinity. Thus the form $V_0^{(r)^2}$ and $V^{(\phi)}$ is consistent with the law of conservation of energy in their Newtonian limits. As $V_0^{(r)^2}$ has to be positive every where equation (5.2) indicates that A^2 should be smaller than the minimum valuable of $\frac{2R}{\left(1 - \frac{2}{R}\right)}$ which yield $|A| < 4$.

References

1- Bardeen, J. M. :1972, Black holes Les Houches (eds) de Witt and de Witt (Gordon and Breach.}

2- Bardeen, J. M. and Press W.H. : 1973 , J. Math. phys **14,**7.

3- Kerr, R.P :1963, phys. Pprev. Lett. **11,** 237.

4- Mishra, K. N. and Chakraborty D.K. : 2003 Astrophysics and space science **286,** 437.

5- Misner, C.W., Thorne K.S. and Wheeler J. : 1972 Gravitation, (Freeman and Co.).