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Orbits of close binary stars with a conservative exchange of mass

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A widely used method for determining the change in the semimajor axes of the circular orbits of close binary stars with conservative exchange was shown to be based on the known Mestschersky–Levy-Civita problem unsuitable for the description of jet streaming of matter through the vicinity of a special point in the Roche lobes. A method of numerical integration of the Mestschersky equations is proposed. The obtained results are compared with those currently in use.

Keywords: Close binary stars; Conservative exchange of mass; Roche lobe; Angular momentum integral

1 Introduction

The orbits of stars in close binary systems with a conservative exchange of mass began to be studied in 1960s. A cycle of works by Kruszevski [1], Hadjidemetriou [2], Piotrowski [3] and other authors, by numerically integrating the equations of a restricted three-body problem, assessed the change of the semimajor axis of stars' relative orbit depending on various perturbation factors.

Subsequently, deduction of the dependence of the semimajor axes on time was substantially simplified by carrying it out for the problem of two bodies with variable masses, assuming the existence of the angular momentum integral:

$$\mathbf{J} = \mathbf{R}_1 \times M_1\mathbf{V}_1 + \mathbf{R}_2 \times M_2\mathbf{V}_2 = \text{const}, \tag{1}$$

where $M_i, \mathbf{R}_i, \mathbf{V}_i$ $(i = 1, 2)$ are, respectively, masses, radius-vectors and velocities of the stars, and the sum of the masses always remains constant $M_1 + M_2 = M = \text{const}$. The grounds for this conclusion were given in [4] and [5].

A strict mathematical deduction of integral (1) for a close binary system has not been given. The existence of the integral has been explained by general considerations on the closeness of the considered mechanical system (i.e., by the absence of external forces) and invariability of total stellar mass.

However, as this work shows, the assumed existence of the angular momentum integral $\mathbf{J}$ is not consistent with the pronounced conservative jet streaming of matter

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through the vicinity of a special point L_1 of the Roche lobe. Therefore, determining the motion of stars in close binary systems with a conservative exchange of mass by means of integral (1) is incorrect.

This work proves the non-existence of integral (1) for a jet exchange of mass through the vicinity of the special point L_1. We show that in the problem of two bodies with variable masses the integral does exist, but for an absolutely different (and, therefore, unsuitable) scheme of mass streaming, when the outflow or inflow of mass occurs at a relative velocity equal to that of a star, but directed oppositely, the way it is done in the Mestschersky–Levy-Civita problem.

Stellar motions are proposed to be determined by a method using the Mestschersky equations.

2 Non-existence of integral (1) for close binary stars with conservative exchange

2.1 Closed and non-closed systems of bodies with variable masses

We shall begin to study the motion of bodies with variable masses from the very beginning, from the second law of mechanics. For bodies with constant masses, the law is usually formulated as follows: the product of the mass of a moving body by its acceleration is equal to the resultant of external forces. For bodies with variable masses, this definition is wrong, and we should use the original definition given by Newton: the velocity of a change of momentum is equal to the resultant of external forces. Systems of bodies with the zero resultant of external forces are referred to as closed. For closed systems, the angular momentum integral is deducible at once or else does not exist.

A known example of a non-closed system is a rocket with the working jet engine in a forceless field. The equation of motion of a rocket is written down as:

$$\frac{d}{d\,t}(m\mathbf{v}) = \dot{m}\mathbf{u} \tag{2}$$

or as

$$m\frac{d\mathbf{v}}{d\,t} = \dot{m}\mathbf{w}, \tag{3}$$

where m and $\mathbf{v}$ are the mass and velocity of motion of a rocket, $\mathbf{u}$ is absolute velocity, and $\mathbf{w} = \mathbf{u} - \mathbf{v}$ is the relative velocity of the efflux from the nozzle of the engine.

It is seen from equations (2) and (3) that no momentum and angular momentum integrals exist in this problem.

2.2 Gylden–Mestschersky problem

The differential equations of motion of bodies with variable masses were first used by Gylden [6], who wrote them down in the same form as for constant masses. However, the strict substantiation of this kind of differential equations was given by Mestschersky [7], who showed that these equations are a special case of more general equations (called now the Mestschersky equations), when the resultant of the jet forces is equal

to zero (e.g., for isotropic mass outflows). For this reason, the problem of the motion of bodies with variable masses and with zero resultant jet forces is called the Gylden–Mestschersky problem.

The equations of motion of the Gylden–Mestschersky problem for two mutually gravitating stars S_1 and S_2 with variable masses are written down as:

$$M_1\frac{d\mathbf{V}_1}{dt}=G\frac{M_1M_2}{R^3}\mathbf{R},\qquad M_2\frac{d\mathbf{V}_2}{dt}=-G\frac{M_1M_2}{R^3}\mathbf{R}, \tag{4}$$

where G is the universal gravitational constant, and $\mathbf{R}=\mathbf{R}_2-\mathbf{R}_1$ is the relative radius-vector of S_2 relative to S_1.

Although the sum of forces in the right-hand side parts of equations (4) is equal to zero, this system remains a non-closed mechanical system, as external forces in problems with variable masses should be determined from equations written down in conformity with the original definition of the second law of mechanics given by Newton. Indeed, re-writing equations (4) as

$$\frac{d}{dt}(M_1\mathbf{V}_1)=\frac{GM_1M_2}{R^3}\,\mathbf{R}+\dot{M}_1\mathbf{V_1},\qquad \frac{d}{dt}(M_2\mathbf{V}_2)=-\frac{GM_1M_2}{R^3}\,\mathbf{R}+\dot{M}_2\mathbf{V_2}, \tag{5}$$

we can notice that the resultant of external forces differs from zero:

$$\frac{GM_1M_2}{R^3}\,\mathbf{R}+\dot{M}_1\mathbf{V}_1-\frac{GM_1M_2}{R^3}\,\mathbf{R}+\dot{M}_2\mathbf{V}_2=\dot{M}_1\mathbf{V}_1+\dot{M}_2\mathbf{V}_2=\dot{M}_2\mathbf{V}\neq 0, \tag{6}$$

where $\mathbf{V}=\dot{\mathbf{R}}=\mathbf{V}_2-\mathbf{V}_1$, $\dot{M}_1=-\dot{M}_2$.

Therefore, the mechanical system of stars S_1 and S_2 is non-closed, and equations (4) are not integrated and do not admit the existence of integrals of momentum, angular momentum and integrals of motion of the centre of masses. But we can write down the quasi-integrals of motion of the centre of inertia C as [8]:

$$\begin{aligned} M\mathbf{V}_C&=M\mathbf{V}_B-\dot{M}_2\,\mathbf{R}-\mathbf{u}_1=\mathbf{a},\\ M\mathbf{R}_C&=m\mathbf{R}_B-\mathbf{u}_2=\mathbf{a}t+\mathbf{b}, \end{aligned} \tag{7}$$

where $\mathbf{R}_B=(M_1\mathbf{R}_1+M_2\mathbf{R}_2)/M$ and $\mathbf{V}_B=(M_1\mathbf{R}_1+M_2\mathbf{R}_2+\dot{M}_2\,\mathbf{R})/M$ is the radius-vector of the centre of masses B and of the velocity of its motion, $\mathbf{u}_1=\int\dot{M}_2\,\mathbf{V}\,dt$, $\mathbf{u}_2=\int\dot{M}_2\,\mathbf{R}\,dt+\int\int\dot{M}_2\,\mathbf{V}\,dt\,dt$, $\mathbf{a}$ and $\mathbf{b}$ are arbitrary vector constants of integration, $M=M_1+M_2=\text{const}$, $\dot{M}_1=-\dot{M}_2$.

From here we can see that the equations of absolute motion of the Gylden–Mestschersky problem (4) do not admit any first integral. Integrals of momentum and angular momentum do not exist in this problem. At the same time, the equation of relative motion of this problem is integrated up to the end. Indeed, the equation of relative motion

$$\frac{d\mathbf{V}}{dt}=-\frac{GM}{R^3}\mathbf{R} \tag{8}$$

coincides with the equation of relative motion in the problem of two bodies with constant masses and, hence, is completely integrated, determining the unperturbed

Keplerian motion of stars. Therefore, the semimajor axis of a relative orbit of stars in the Gylden–Mestschersky problem always remains constant.

It is seen from this consideration that the Gylden–Mestschersky problem as an independent model is not suitable for determining the change of the semimajor axis of the orbits of stars because it fails to explain how outflowing particles of matter move from one star on the other.

However, this problem can be conveniently used as unperturbed relative motion of stars when deriving equations of perturbed motion.

2.3 Mestschersky–Levy-Civita problem

Let us consider now the known model investigated first by Mestschersky [7] and then by Levy-Civita [9]. This problem called the Mestschersky–Levy-Civita problem is the only model for two bodies with variable masses, in which the angular momentum integral does exist.

However, the Mestschersky –Levy-Civita problem cannot be used for a close binary system of stars. Indeed, the equations of motion in the Mestschersky–Levy-Civita problem are written down as

$$M_1\frac{d\mathbf{V}_1}{d\,t} = G\frac{M_1\,M_2}{R^3}\mathbf{R} - \dot{M}_1\mathbf{V}_1, \qquad M_2\frac{d\mathbf{V}_2}{d\,t} = -G\frac{M_1\,M_2}{R^3}\mathbf{R} - \dot{M}_2\mathbf{V}_2 \tag{9}$$

or, in a different form, as

$$\frac{d}{d\,t}(M_1\mathbf{V}_1) = G\frac{M_1\,M_2}{R^3}\mathbf{R}, \qquad \frac{d}{d\,t}(M_2\mathbf{V}_2) = -G\frac{M_1\,M_2}{R^3}\mathbf{R}. \tag{10}$$

The equation of relative motion in the Mestschersky–Levy-Civita problem has the form:

$$\frac{d\mathbf{V}}{dt} = -\frac{GM}{R^3}\mathbf{R} - \dot{M}_2\left(\frac{\mathbf{V}_1}{M_1} + \frac{\mathbf{V}_2}{M_2}\right). \tag{11}$$

From equations (10), we can see that this system is closed, as the resultant of external forces is equal to zero. For this reason, the Mestschersky–Levy-Civita problem admits integral (1).

But, as we can see from (9), the outflow of matter from star S_1, as well as its inflow to star S_2, occur with relative velocities $\mathbf{W}_1 = -\mathbf{V}_1$ and $\mathbf{W}_2 = -\mathbf{V}_2$, respectively, while in a close binary system the outflow of matter from star S_1 occurs through the vicinity of the special Euler point L_1 towards the other star with the module of relative velocity by an order of magnitude smaller than the orbital velocity of star S_1. The velocity of inflow to S_2 also differs from velocity $-\mathbf{V}_2$ both by size and direction.

It is for this reason that the Mestschersky–Levy-Civita problem is not suitable for the description of motion of close binary stars.

In astronomy, the Mestschersky–Levy-Civita problem is, e.g., used to study motions of stellar systems or separate stars in a motionless dust cloud or a nebula with sticking of matter to moving stars.

3 Mestscherky equations

Motions of stars of a close binary system with a conservative exchange of mass should be studied with the help of Mestschersky equations. Let us consider as external forces (except for forces of mutual gravitation) two jet forces $\dot{M}_1\mathbf{U}_1$ and $\dot{M}_2\mathbf{U}_2$ acting on stars S_1 and S_2, respectively. The designations used here are $\mathbf{U}_1$ (absolute velocity of outflow from S_1) and $\mathbf{U}_2$ (absolute velocity of inflow to S_2).

Then the equations of motion will be written down as

$$\frac{d}{d\,t}(M_1\mathbf{V}_1) = G\frac{M_1\,M_2}{R^3}\mathbf{R} + \dot{M}_1\mathbf{U}_1, \qquad \frac{d}{d\,t}(M_2\mathbf{V}_2) = -G\frac{M_1\,M_2}{R^3}\mathbf{R} + \dot{M}_2\mathbf{U}_2. \quad (12)$$

As $\mathbf{U}_1 \neq \mathbf{U}_2$, then, in contrast with the Mestschersky–Levy-Civita problem, in which $\mathbf{U}_1 = \mathbf{U}_2 = 0$, equations (12) do not admit any angular momentum integral.

The absolute velocities $\mathbf{U}_i$ of outflow or inflow can be expressed through relative $\mathbf{W}_i$ and translational $\mathbf{V}_i$ velocities of stellar motion:

$$\mathbf{U}_1 = \mathbf{W}_1 + \mathbf{V}_1, \qquad \mathbf{U}_2 = \mathbf{W}_2 + \mathbf{V}_2. \quad (13)$$

Then equations (12) can be rewritten as:

$$M_1\frac{d\mathbf{V}_1}{d\,t} = G\frac{M_1\,M_2}{R^3}\mathbf{R} + \dot{M}_1\mathbf{W}_1, \qquad M_2\frac{d\mathbf{V}_2}{d\,t} = -G\frac{M_1\,M_2}{R^3}\mathbf{R} + \dot{M}_2\mathbf{W}_2. \quad (14)$$

From (14), we have the equation of relative motion

$$\frac{d\mathbf{V}}{dt} = -\frac{GM}{R^3}\mathbf{R} + \dot{M}_2\left(\frac{\mathbf{W}_1}{M_1} + \frac{\mathbf{W}_2}{M_2}\right), \quad (15)$$

which makes use of the condition of conservative exchange: $\dot{M}_1 = -\dot{M}_2$. Without the loss of generality, we can consider $\dot{M}_2 > 0$.

Equations (12), (14) or (15) can be used to determine stellar motions only at the known outflow and inflow velocities.

Similar to the efflux of gas from a cylinder through a small aperture occurring with velocity close to that of sound, the velocity of outflow $\mathbf{W}_1$ from star S_1 can be considered to be close to the velocity of sound in the atmosphere of star S_1. This velocity is several percent of the orbital velocity $\mathbf{V}_1$ [10].

To determine velocity $\mathbf{W}_2$, we should know the trajectories of motion of flowing particles from the special Euler point L_1 to the surface of star S_2, which can be considered to be a sphere of radius $R^{(2)}$. These trajectories can be obtained by numerical integration of the equations of motion of flowing particles, using for this purpose the restricted three-body problem (see Figs. 1 and 2).

The constant presence of a jet of flowing matter leads to the emergence of two more forces of attraction of stars by a jet, considering which, aquations (14) will be rewritten as

$$\begin{aligned} M_1\frac{d\mathbf{V}_1}{d\,t} &= G\frac{M_1\,M_2}{R^3}\mathbf{R} + \dot{M}_1\mathbf{W}_1 + G\frac{M_1\,M_3}{R_{13}^3}\mathbf{R}_{13}, \\ M_2\frac{d\mathbf{V}_2}{d\,t} &= -G\frac{M_1\,M_2}{R^3}\mathbf{R} + \dot{M}_2\mathbf{W}_2 + G\frac{M_2\,M_3}{R_{23}^3}\mathbf{R}_{23}, \end{aligned} \quad (16)$$

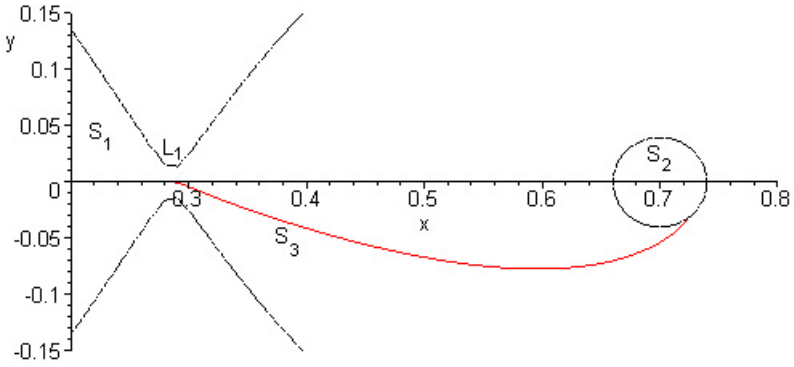


Figure 1 The trajectory of streaming particles at $q = M_2/M_1 = 3/7$.

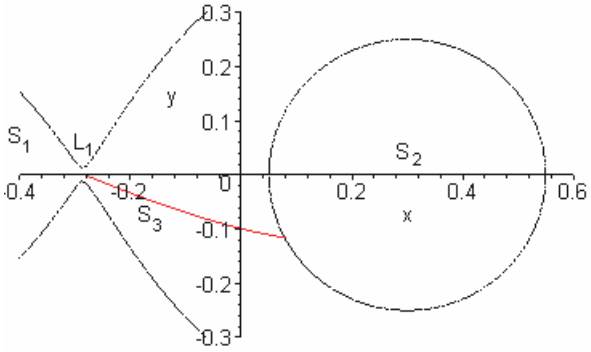


Figure 2 The trajectory of streaming particles at $q = M_2/M_1 = 7/3$.

and equation of relative motion (15) as

$$aa\frac{d\mathbf{V}}{dt} + \frac{GM}{R^3}\mathbf{R} = \mathbf{F}, \tag{17}$$

where

$$\mathbf{F} = \dot{M}_2\left(\frac{\mathbf{W}_1}{M_1} + \frac{\mathbf{W}_2}{M_2}\right) + GM_3\left(\frac{\mathbf{R}_{23}}{R_{23}^3} - \frac{\mathbf{R}_{13}}{R_{13}^3}\right), \tag{18}$$

$M_3 = \dot{M}_2\tau$ is the mass of a jet, τ is the time of transference of flow particle from point L_1 to the surface of S_2; $\mathbf{R}_{13}$ and $\mathbf{R}_{23}$ are the radius-vectors of the centre of masses of a jet relative to stars S_1 and S_2, respectively.

Jet forces and forces of attraction by a jet are very small in comparison with forces of mutual gravitation between stars. But their action is of secular character, and at large temporal intervals these small forces lead to significant perturbations of orbital stella elements.

Equation (17) is, certainly, not integrated and does not admit the existence of the angular momentum integral and other integrals.

The right-hand side of equation (17) contains a small parameter $\dot{M}_2$ as a multiplier. If this multiplier is put to be equal to zero, i.e., if the influence of jet forces and forces

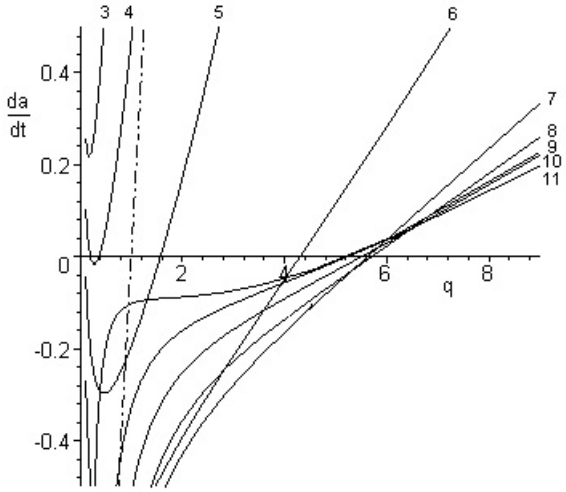


Figure 3 The change of the semi-major axis derivative da/dt in mass outflow. For notations, see Fig. 5; the dash and dot curve corresponds to relation (22) at $a = 1$.

of attraction of the stars by a jet is neglected, equation (17) is transformed to the equation of relative motion for the Gylden–Mestschersky problem, according to which all Keplerian elements of the relative orbit remain constant.

Thus, relative motions of stars can be determined using equation (17), the right-hand side $\mathbf{F}$ of which represents a perturbation with respect to the known Keplerian unperturbed motion in the Gylden–Mestschersky problem.

Equation (11), which is currently used to determine the evolution of the circular orbits of stars in close binary systems, is obtained from equations (17) and (18), if we put velocities of the outflow $\mathbf{W}_1$ and inflow $\mathbf{W}_2$ to be equal to velocities of stellar motion taken with opposite signs and, besides, the attraction of the stars by a jet is excluded from consideration, e.g., by putting $\tau = 0$.

Following the Lagrangian method of the variation of arbitrary constants, the equations of perturbed motion can be written down in osculating elements [11]. Restricting ourselves to the consideration of only circular orbits of stars and only one osculating element, the semimajor axis a, we obtain the equation

$$\dot{a} = \frac{2}{\Omega} T, \tag{19}$$

where $\Omega = \sqrt{GM}/a^{3/2}$ is the angular velocity of the circulation of the stars on an orbit, and $T = [\mathbf{F}]_T$ is the transversal component of perturbed acceleration:

$$T = \left[\dot{M}_2 \left(\frac{\mathbf{W}_1}{M_1} + \frac{\mathbf{W}_2}{M_2} \right) + GM_3 \left(\frac{\mathbf{R}_{23}}{R_{23}^3} - \frac{\mathbf{R}_{13}}{R_{13}^3} \right) \right]_T . \tag{20}$$

The equation for the osculating semimajor axis in the Mestschersky–Levy-Civita problem can be transformed to the same form (17), but the right-hand side of $\mathbf{F}$ will be different:

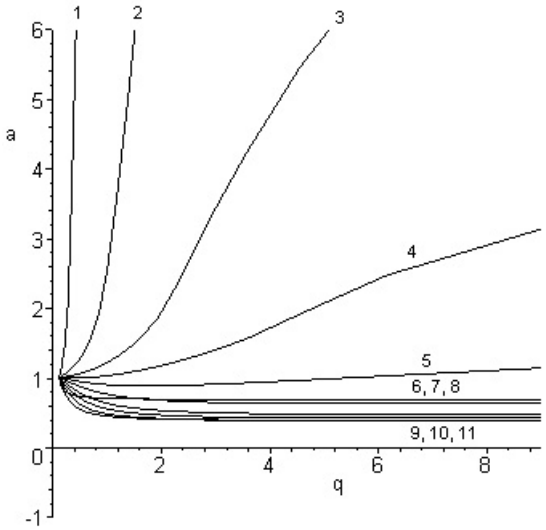


Figure 4 The change of the semi–major axis a of the relative orbit of stars in mass outflow. Notations: 1, $R_0^{(2)}/a_0 = 0.0600$; 2, $R_0^{(2)}/a_0 = 0.0730$; 3, $R_0^{(2)}/a_0 = 0.0750$; 4, $R_0^{(2)}/a_0 = 0.0758$; 5, $R_0^{(2)}/a_0 = 0.0765$; 6, $R_0^{(2)}/a_0 = 0.0778$; 7, $R_0^{(2)}/a_0 = 0.0800$; 8, $R_0^{(2)}/a_0 = 0.0820$; 9, $R_0^{(2)}/a_0 = 0.0878$; 10, $R_0^{(2)}/a_0 = 0.1$; 11, $R_0^{(2)}/a_0 = 0.2$.

$$\mathbf{F} = -\dot{M}_2 \left(\frac{\mathbf{V}_1}{M_1} + \frac{\mathbf{V}_2}{M_2} \right). \tag{21}$$

After simple transformations, we can obtain from (21) an equation for the osculating semimajor axis in the form:

$$\dot{a} = 2a\dot{M}_2 \left(\frac{1}{M_1} - \frac{1}{M_2} \right), \tag{22}$$

which has been generally used for half a century now to establish the dynamic evolution of stellar orbits in close binary systems. However, this equation, as we show above, does not correspond to the true physical process of streaming in binary systems.

In particular, this formula does not take into account the influence of jet forces on the stellar orbit. The change of the semimajor axis of an orbit strongly depends on radius $R^{(2)}$ of an accepting star S_2, which is seen from Figs. 1 and 2, whereas in the Mestschersky–Levy-Civita problem there is no dependence like this.

Figures 3 and 4 present a comparison of the change of the semimajor axis a and its derivative $\dot{a}$, which are determined by formulas (19), (20) and (22) using the method of numerical integration.

Figure 5 shows a diagram obtained by numerical integration, which presents the areas of positive and negative values for the derivative of the semimajor axis $\dot{a}$ (1).

Figures 1–5 were adopted from [12].

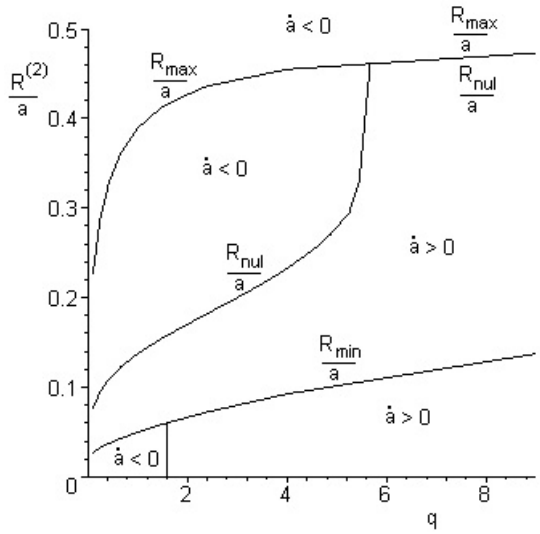


Figure 5 A diagram of the types of conservative mass exchange in a binary stellar system. Notations: R_{max}/a, the boundary of contact system formation; R_{min}/a, the boundary of accretion disk formation; R_{nul}/a, the boundary of the change of sign for the derivative $\dot{a}$ at a (semidivided) stream conservative mass exchange. The regions where $\dot{a} > 0$ and $\dot{a} < 0$ are shown.

4 Conclusion

Equation (22) obtained from the angular momentum (1) and used for over half a century now is valid for the Mestschersky–Levy-Civita problem. However, this problem does not reflect a true physical process, the pronounced jet streaming of matter through the vicinity of the special point L_1 of the Roche lobe. In the Mestschersky–Levy-Civita problem, the outflow of matter from star S_1 occurs in the direction opposite to the velocity of motion and with a value equal to the velocity, i.e., $\mathbf{W}_1 = -\mathbf{V}_1$. Similarly for the inflow of matter to the other star $\mathbf{W}_2 = -\mathbf{V}_2$.

For this reason, the Mestschersky–Levy-Civita problem should be acknowledged to be unsuitable for the description of relative motions of stars with a conservative exchange of mass.

This work used the method of numerical integration of Mestschersky equations to determine the change of the semimajor axis of stellar orbits with account of jet forces and forces of attraction of stars by a jet.

The obtained results should not be considered to be final, as the values of jet forces and forces of attraction of a jet could be refined. Other forces not considered in this work could also be taken into account.

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