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Morse–Smale complexes and graphodynamics of active regions on the Sun

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In this paper we propose to use the Morse–Smale topological complexes (MS) as dynamic models of the magnetic fields in the solar active regions (AR). The complexes are based on the gradient representation of the field (according to Forman) in the framework of the discrete Morse theory. The MS complexes are formed by a network of cells, each of which contains a maximum, a minimum and two saddle points. This network can be considered as a graph and it allows a persistent simplification. The evolution of the AR leads to a graph rearrangement, which is referred to as graphodynamics. It is clearly described by the spectrum of a discrete Laplacian. The article gives an example of the application of this technique to the solar HMI/SDO magnetograms.

Keywords: Morse–Smale complex, Sun: activity, magnetic fields, Sun: magnetic topology

1 Introduction

The dynamics of magnetic fields in Active Regions (AR) on the Sun is tracked by changing the geometry and topology of the observed patterns in solar magnetograms. The characterization of dynamic regimes requires appropriate descriptors to quantify the spatial and temporal complexity of digital images. The most popular descriptors are based on a rich observational experience and quantify magnetograms in some way. Physical values represented in a discrete form served as a basis for such descriptors. The correctness of this representation is usually not discussed.

There are about 30 descriptors that also cover vector magnetograms (Barnes, 2007). A high dimension of attribute space leads to the so-called "curse of dimensionality". The points representing the descriptors are concentrated in the thin shells of hypersurfaces, and the usual clustering algorithms are no longer effective. A number of the descriptors have recently been reduced to 18 SHARP-parameters (Bobra, 2014)

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that have been chosen according to their prospects for the prediction of solar flares. But this approach is burdened with the effects of concentration of measure. Thus, any predictor that depends on the influence of many independent variables (but not too much on any of them) is essentially constant (Talagrand, 1996).

There is another approach to the problem of dynamic model construction. It is an appropriate global discrete model construction, for example, a graph or a skeleton. A graph consists of the vertex set $\{V_i\}$, and the set of edges $\{E_i\}$, which are pairs of vertices. There are known examples of the use of graphs in solar physics (Aschwanden, 2010). In (Schrijver, 1997), photospheric field structure was described using Voronoi diagrams and Delaunay triangulation. These discrete structures are also used in automatic detection of AR (Turmon, Mukhtar, 1997).

Consider a solar magnetogram as the expression of the solar magnetic field. It is convenient to use critical points of the magnetogram data as vertices of the graph. Note that such critical points play an important role in solar flares description and prediction problems (Wang, Wang, 1996). One notable example of a topological graph is the Čech complex, or the nerve of a covering. In (Knyazeva, 2015), the dynamic regimes of AR are tracked by the HMI/SDO magnetogram data using the Čech complex. Another example is a critical net (Gu, 2010). Vertices of a critical net are maxima and minima (not saddles) of the magnetogram data. In a critical net, the stable minima are connected to the closest maxima in the scale-space (Knyazeva, 2016).

However, the most interesting approach for the modeling of dynamic systems is the Morse–Smale complex, which represents a gradient structure of the field (Allili, Corriveau, 2007). Each cell of the 2-dimensional MS complex contains one maximum, one minimum, and two saddle points. MS complexes can be interpreted as the discrete multivalued dynamic system (Allili, 2007). There is a known MS complex simplification technique based on the concept of persistence (Günther, 2014). The first results on editing of solar magnetograms with MS complexes are presented in our work (Makarenko, 2016).

The variation of the Morse graph describes the dynamic regimes of the magnetic field in the AR. This variation is often called graphodynamics (Ayzerman, 1997, Bradley, Stuart, 1998).

Each graph has the correspondent discrete Laplacian, or the Kirchhoff matrix (Molitierno, 2016). The dynamic properties of the graph may be expressed in the terms of the spectral graph theory (Chung, 1997). Thus, the Morse graph and its spectrum provide a useful tool for description of dynamic regimes of the AR magnetic field.

The goal of this work is:

- to provide a short survey of MS complex construction technique, and the basic concepts of the spectral graph theory;
- using these tools, to give a characterization of the solar magnetic field dynamic regimes for a time series of LOS-magnetograms in the case of flare AR.

In sections “Morse theory” and “Discrete Laplacian and its spectrum” we provide a theoretical background of the considered topics. After that, we present the results of the graphodynamic analysis of the active region AR12673.

2 Morse theory

Let f be a smooth real-valued function defined on a smooth compact 2-dimensional manifold $\mathbf{M}$. Consider ∇f, a gradient field of f. A curve γ on $\mathbf{M}$ that in every point $p \in \gamma$ has a tangent vector $\nabla f(p)$ is called an integral curve. Points $\lim_{t\to-\infty}\gamma(t)$ and $\lim_{t\to+\infty}\gamma(t)$, if these limits exist, are called origin and destination of the integral curve $\gamma(t)$.

Moreover, if all critical points (minima, saddles, and maxima) of f are non-degenerate, then f is called a Morse function. The integral lines of a Morse function f obeys certain properties (Sousbie, 2011):

- Each integral has origin and destination in a critical point.
- There exists a unique integral curve passing through each point $p \in \mathbf{M}$; two distinct integral curves are not intersect (but may share origin and / or destination). An integral curve passing through a critical point degenerates to this point.
- The set of all integral curves covers all of $\mathbf{M}$ and induces an equivalence relation on $\mathbf{M}$: $p, p' \in \mathbf{M}$ are equivalent iff integral curves γ_p and $\gamma_{p'}$ passing through p and p' have the same origin and destination.

The above relation defines a partition of $\mathbf{M}$ into cells, called a Morse–Smale complex (MS complex).

The integral curves connecting maxima with saddles, and saddles with minima, form a 1-dimensional skeleton of a MS complex. Such curves are called arcs of a MS complex.

Now we introduce a notion of topological persistence, which we use for simplification of MS complex and underlying function f.

Let $\mathbf{M}_t = f^{-1}((-\infty; t])$ be a sublevel set of f. When t increases, topology of $\mathbf{M}_t$ may change. By (Milnor, 1963), topology of $\mathbf{M}_t$ changes at level t iff $f(p) = t$ for some critical point p.

In 2-dimensional case, the only topological features are connected components (c. c.) and holes. If a topological feature (c. c. or hole) appears at level t_0, then we say that this feature "born at the time t_0". Also if a topological feature vanishes (one c. c. joins another one, or hole is contracted) at level t_1, then we say that this feature "dies at the time t_1". The "lifetime" $t_1 - t_0$ of a topological feature is called persistence.

The mean concept of the topological simplification is to eliminate less significant topological features (with small persistence values), and retain features with large persistence.

Persistence associates a pair of critical points with a topological feature — a point that "gives birth" to this feature, and a point that "kills" it. These pairs of critical points are called persistent pairs.

The topological simplification step consists of two steps: (i) find a persistent pair with the smallest persistence value, and (ii) change the function f in a way that the result will contain all critical points, instead of the points in this pair. Generally, it is true that the points in this pair are connected by arc of the MS complex, so we can change f locally (by changing function values on arc and neighbouring cells) in the way that two critical points vanish, and other critical points remain the same.

We perform simplification steps until we reach user-defined threshold, or until we attain the minimal number of critical points (for example, the function on a sphere has at least two critical points, and the function on a torus has at least four).

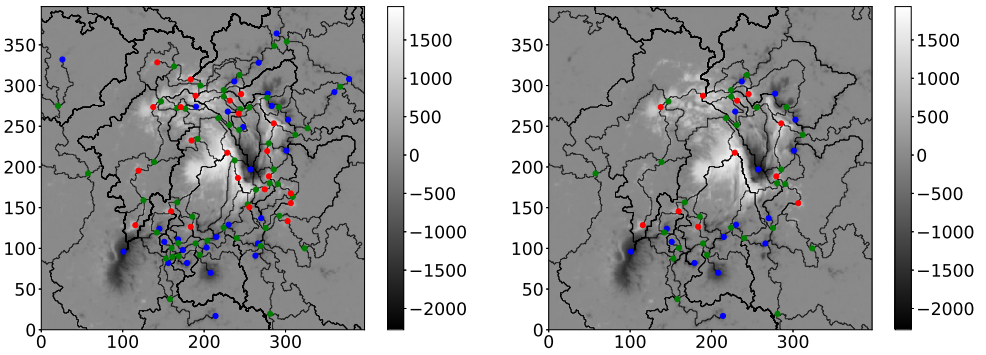


Figure 1 The MS complexes of the solar magnetogram, AR 12673, at the different levels of simplification. Minima are depicted by blue points, saddles – by green points, and maxima – by red points.

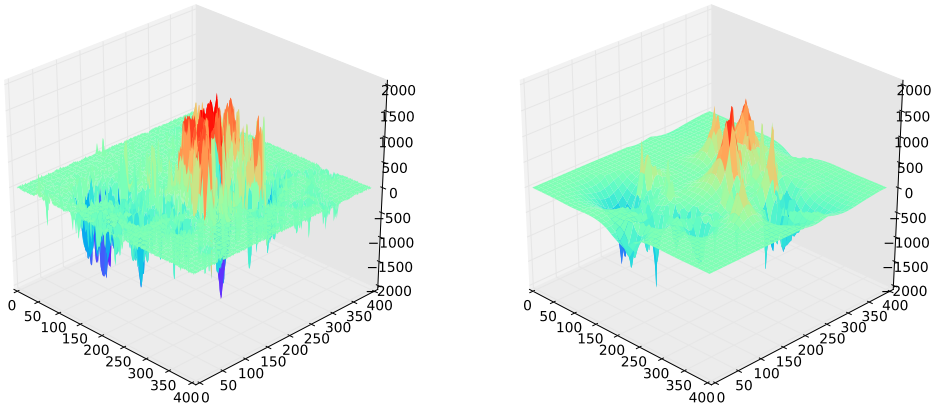


Figure 2 3D plots of the original function (on left) and the result of topological simplification (on right). Small fluctuations of the magnetic field are smoothed, and the significant critical values are retained.

3 Discrete Laplacian and its spectrum

Let $G = (V, E)$ be a simple graph on n vertices, $V = \{1, 2, \ldots, n\}$. An adjacency matrix of G is $n \times n$ matrix A such that:

$$A_{ij} = \begin{cases} 1, \text{ if } (i,j) \in E; \\ 0, \text{ otherwise.} \end{cases}$$

If $(i,j) \in E$, then vertices i and j called adjacent. Number of vertices, which are adjacent to vertex i, is called degree of i, and is denoted by $\deg i$.

A degree matrix of G is $n \times n$ matrix D such that:

$$D_{ij} = \begin{cases} \deg i, \text{ if } i = j; \\ 0, \text{ otherwise.} \end{cases}$$

A Laplacian matrix of G is defined as:

$$L = D - A.$$

The elements of L are given by

$$L_{ij} = \begin{cases} \deg i, \text{ if } i = j; \\ -1, \text{ if } i \neq j \text{ and } i \text{ is adjacent to } j; \\ 0, \text{ otherwise.} \end{cases}$$

A set $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$ of eigenvalues of L is called the spectrum of Laplacian. λ_0 is always 0. The number λ_1 is nonzero for a connected graph, and is called the algebraic connectivity of a graph G.

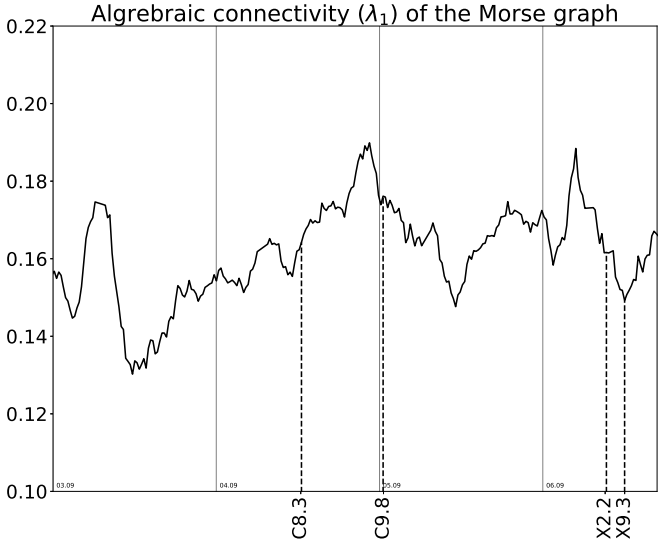


Figure 3 Algebraic connectivity of the Morse graph of AR12763 magnetograms (September 3–6, 2017). One day before the flare activity on the Sun, the algebraic connectivity value decreased to 0.13. Classes of flares are marked on the x-axis.

4 Results

For the numerical experiments, we used LOS magnetograms of solar active region AR12673. In September, 2017 it was the most active flaring region of Cycle 24 (Sun, Norton, 2017). It produced 4 flares above X1 class, 8 flares above M3 class, and X-9.3 (SOL2017-09-06T11:53). As an example, on Fig. 1 (a, b), two consequent MS complexes of one magnetogram with different thresholds of the persistence simplification are presented.

On Fig. 2, 3D-plots of the initial and the simplified magnetograms are shown. The MS complex of a simplified magnetogram is the same as on Fig. 1 (b).

For each magnetogram, the spectrum of the discrete Laplacian of the Morse graph was calculated. For the flaring activity diagnostics, we used the most significant spectrum value, a so-called algebraic connectivity λ_1, the second minimal value ($\lambda_0 = 0$). Its variation at the period September 3–6, 2018, is shown on Fig. 3. We can see a decrease of the algebraic connectivity one day before a flaring activity period, and an increase right before the flares.

A rigorous assessment of this phenomenon on a well-founded sample is surely needed. Our goal in this paper was a description of the contextless way of magnetic dynamics tracking in AR, in the framework of the topological invariants.

Conclusion

In this paper, the methods of the discrete Morse theory are considered to analyse the dynamics of solar flare active regions (ARs). We build cell CW complexes for the line of sight of HMI / SDO magnetograms. In order to remove the noise effect, i.e. cells formed by small values of the field at singular points, we implement the procedure of a persistent simplification of the complex. The evolution of the AR induces a rearrangement of the graph caused by the emergence of new magnetic elements. Considering the CW complex as a graph, we compute the eigenvalue spectrum of the discrete Laplacian. It allows us to "hear" the graph and gives a quantitative description of graphodynamics for the time sequence of magnetograms and the corresponding CW complexes. We obtained the evolution of the second eigenvalue (λ_1) of Laplacian for the graphodynamics of ARs. Comparing the variation of this parameter with the flare efficiency suggests that the proposed approach may be useful for AR flare forecasting. The eigenvalues of the discrete Laplacian are related to the Betti numbers and the discrete Ricci curvature. Therefore, the approach allows us to describe dynamic regimes by invariant topological characteristics.

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