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On the restricted four-body problem when the primaries are triaxial rigid bodies

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This paper deals with the equilateral restricted four-body problem (ER4BP) formulated on the basis of Lagrange's triangular solutions. We have further supposed that all the three primaries situated on the vertices of an equilateral triangle are triaxial rigid bodies with equal masses. The condition under which the equilateral triangular configuration remains maintained and the equations of motion have been deduced.

Keywords: ER4BP, triaxial rigid body, Lagrangian configuration

1 Introduction

In the present days the restricted four-body problem (R4BP) is studied by many scientists. The problem describes the motion of the fourth body with infinitesimal mass, moving under the gravitational effects of three finite bodies, called primaries, which move in circular orbits around their common center of mass on account of their mutual gravitational attraction. The infinitesimal mass does not influence the motion of the primaries. The problem of restricted four bodies has fascinated many researchers and scientists and very significant results have been produced by them in an effort to unveil and predict the motion of celestial bodies. Analogous to the classical restricted three-body problem, we consider the classical restricted four-body problem (CR4BP) when the primaries are point masses or spherical in shape. Further, it is sensible to modify the model of CR4BP by taking the different shape of the primaries and other perturbing forces like radiation pressure, P-R drag effects etc. The oblateness or triaxiality of a body can produce perturbations or deviations from the two-body motion. The orbit of the fifth satellite (Amalthea) of Jupiter is one of

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the most prominent examples of perturbations arising from oblateness in the solar system (Moulton (1914)).

In recent times, many authors like Arribas et el. (2016a), Kalvouridis et al. (2006a), Kalvouridis et al. (2006a), Asique et al. (2015b), Papadouris and Papadakis (2013), Singh and Vincent (2015a), Singh and Vincent (2016) etc. have studied the R4BP by including the radiation pressure force. Other authors (Kumari and Kushwah (2013), Asique et al. (2015a) etc.) have included the oblateness of the primaries in the R4BP.

Recently, a series of studies exist on the R4BP in the equilateral configuration: Alvarez-Ramírez and Vidal (2009) studied the equilateral restricted four-body problem by taking the masses of primaries equal. Baltagiannis and Papadakis (2011a) have studied numerically the problem of four bodies, three of which called primaries are finite and moving in circular orbit around their common center of mass fixed at the origin of the co-ordinate system, according to the solution of Lagrange where they are always at the vertices of an equilateral triangle. Further, they have shown that the existence of collinear and non-collinear libration points and their numbers depend on the mass parameter of the primaries in the problem. They have also shown that the collinear equilibrium solutions do not exist for unequal masses of the primaries. Érdi and Czirják (2013) have studied the central configurations of four bodies with an axis of symmetry. They have provided the complete solution for a symmetric case of central configurations of the planar problem of four bodies, in which two bodies are on an axis of symmetry, and the other two bodies have equal masses are situated symmetrically with respect to the axis of symmetry. They have derived the formulae representing exact analytical solutions of the four-body problem. Suraj and Hassan (2014) have studied the photogravitational version of Sitnikov four-body problem. They have found a solution which is valid for small bounded oscillations in case of moderate eccentricity of the primary. Further, they have linearized the equation of motion to obtain the Hill's type equation. By using the Courant and Snyder transformation, Hill's equation is transformed into harmonic oscillator type equation. They have used the Lindstedt–Poincaré perturbation method and again they have applied the Courant and Snyder transformation to obtain the final result. Alvarez-Ramírez and Barrabés (2015) studied the transport orbits in an equilateral restricted four-body problem by taking two primaries of equal masses. Further, they studied numerically the existence of families of unstable periodic orbits. It is also observed that the homoclinic and heteroclinic connections as well as transit orbits traveling from and to different regions exist due to invariant stable and unstable manifolds.

Alvarez-Ramírez et al. (2016) have studied the nonlinear stability analysis in an equilateral restricted four-body problem. They have calculated the sixth order normal form and constituted the stability of libration points in the case of failure of the fourth order analysis and also discuss the stability at the $2:1$ resonances for the equilibrium point L_3 and the symmetrical L_5 and L_6 points. Arribas et el. (2016b) have studied the out-of-plane equilibria in the symmetric collinear restricted four-body problem with radiation pressure. They have studied three-dimensional case. The three primaries in which the masses and radiation forces of peripheral bodies are equal are supposed to be in a collinear central configuration. Further, they have presented a complete study of the position and stability of the equilibrium points out of xy-plane.

Singh and Vincent (2015c) have studied the effect of perturbations in the Coriolis and centrifugal forces on the stability of equilibrium points in the restricted four-body problem. Further, it is mentioned that the positions of these equilibrium points are affected by a small perturbation in the centrifugal force, but are unaffected from the Coriolis force. Papadakis (2016a) has studied the families of three-dimensional periodic solutions in the circular restricted four-body problem. Since he has studied the simplest $3D$ periodic orbits and he has found 21 such families of simple $3D$ symmetric periodic orbits and stability of each calculated $3D$ periodic orbits have been also discussed. The characteristic curves as well as the stability diagrams of families of $3D$ periodic orbits are also illustrated in his study. Papadakis (2016b) has studied the families of asymmetric periodic solutions in the restricted four-body problem. In this study, he has found series of asymmetric critical orbits for various values of the primaries. From these critical orbits, he has calculated and presented five new families of simple periodic orbits which are asymmetric with respect to the axes. Additionally, he has described a grid method and obtained initial conditions for new asymmetric double-periodic orbits. Further, he has determined ten families of asymmetric double-periodic orbits from the bifurcations of the previous five asymmetric families using the special generating horizontally critical periodic orbits.

Asique et al. (2016a) have studied the R4BP when the third primary is a triaxial rigid body and source of radiation. At most, there are ten non-collinear equilibrium points in total. However, the existence of the number of equilibrium points depends on triaxiality parameters. They have also drawn the zero-velocity surfaces which determine the possible permitted boundary regions. Further, the stability of non-collinear libration points for the various mass parameter, the radiation parameter and the triaxiality of the third body are also studied. Asique et al. (2016b) have further extended the previous study by taking the third primary as an ellipsoid and showed the existence and locations of equilibrium points. They have also discussed the zero velocity curves (ZVCs) for various values of the Jacobian constant. Mittal et al. (2016) have studied the restricted four-body problem with variable mass. They have found that there exist eight equilibrium points, out of which, two are collinear and six are non-collinear. Further, they have shown the significant effect of the Jacobian constant and all other parameters.

The most intrinsic properties of the dynamical system are revealed by the basins of a ttraction associated with the libration points (e.g. Suraj et al. (2017a,b)). Zotos (2016) has studied the basins of convergence in the planar four-body problem and observed that the mass parameter is one of the most important factors in the dynamical system. Zotos (2017) has discussed the escape and collision dynamics in the planar equilateral restricted four-body problem. In this study, he has successfully located the escape and the collision basins and also managed to correlate them with the corresponding escape and collision times of orbits. A thorough numerical analysis has been studied in the configuration (x, y) plane as well as in the (x, y) space. He classifies the initial conditions of orbits into four main categories: the bounded regular orbits, trapped chaotic orbits, escaping orbits and the collision orbits. Interpreting the collision motion as leaking in the phase space, he has related his results to both chaotic scattering and the theory of leaking Hamiltonian systems.

Yi Qi and Shijie Xu (2016) have studied the lunar gravity assist orbits in the restricted four-body problem. The lunar gravity assist orbits starting from the Earth

are studied in the Sun–Earth–Moon–spacecraft system. Further, they have discussed the sphere of influence of the Earth–Moon system. Numerical calculations show that inside the sphere of influence of the Earth–Moon, the effect of the Sun on the lunar gravity assist orbits is quite small. But outside the sphere of influence of the Earth–Moon, the Sun perturbation influences the trend of the lunar gravity assist orbit remarkably.

This paper is prepared as follows: in Section 2 we have discussed how the triaxiality parameter due to primaries influences the equilateral triangle configuration. Section 3 deals with the equations of motion of the infinitesimal mass when the primaries are triaxial rigid bodies. Finally, the paper ends with Section 4 where discussion and conclusions are presented.

2 Condition for existence of equilateral triangle configuration

In the present problem, we have taken three triaxial rigid bodies known as primaries, further, it is supposed that these primaries are symmetrical in all respects with equal masses ($m_1 = m_2 = m_3 = m$). All these primaries are placed at the vertices of an equilateral triangle $P_1P_2P_3$ with side l and are moving around their common center of mass O with constant angular velocity n. Further, taking a rotating frame of reference $Oxyz$ which is centered in the center of rotation of the primaries and revolving around the $z-$ axis with the constant angular velocity of the bodies. The primaries m_1, m_2 and m_3 remain fixed in the rotating frame of reference. We also assume that the axes of the triaxial rigid bodies remain parallel throughout the motion of the synodic co-ordinate system. The line joining O with the center of the primary m_1 is taken as $x-$axis and Oy in the plane of motion of the primaries as $y-$axis and the line passing through O and perpendicular to the plane of motion is taken as $z-$axis. This configuration is shown in Fig. 1. It is well known that the equilateral triangle configuration is possible when all the three masses placed at the vertices of an equilateral triangle are point masses. Here, we have found the condition under which the equilateral triangle configuration remains unchanged throughout the motion when the primaries are triaxial rigid bodies instead of point masses. The co-ordinates of the primaries $m_i, (i = 1, 2, 3)$ at time $t = 0$ are $(x_1, y_1, z_1) = (\frac{l}{\sqrt{3}}, 0, 0)$, $(x_2, y_2, z_2) = (-\frac{x_1}{2}, \frac{l}{2}, 0)$ and $(x_3, y_3, z_3) = (x_2, -y_2, 0)$, respectively.

Let us assume that $\vec{r_i} = r_i\hat{\upsilon}_i$, denote the position vector of three triaxial primaries m_i and $\hat{\upsilon}_i$ are corresponding unit vectors. Since the primaries are moving in circular orbit therefore $r_1 = r_2 = r_3$. Suppose that the axes of the triaxial rigid bodies m_i are a_i, b_i and c_i respectively. Further, $A_i = m_i\frac{b_i^2 + c_i^2}{5}$, $B_i = m_i\frac{a_i^2 + c_i^2}{5}$, $C_i = m_i\frac{a_i^2 + b_i^2}{5}, (i = 1, 2, 3)$ are moments of inertia of the triaxial rigid bodies m_i about the principal axes of the body. Since we have supposed that all the three triaxial primaries m_i are symmetrical in all respects therefore $a_i = a, b_i = b$ and $c_i = c$. Further, the direction cosines of the principal axes of the triaxial rigid bodies m_i are $(\alpha_{1i}, \beta_{1i}, \gamma_{1i}) = (1, 0, 0)$, $(\alpha_{2i}, \beta_{2i}, \gamma_{2i}) = (0, 1, 0)$ and $(\alpha_{3i}, \beta_{3i}, \gamma_{3i}) = (0, 0, 1)$. Let

$(\varrho_1, \varrho_2, \varrho_3)$ and $(\acute{\varrho_1}, \acute{\varrho_2}, \acute{\varrho_3})$ be the direction cosines of the line P_1P_2 relative to the principal axes of m_1 and m_2 respectively. Therefore,

$$\begin{aligned}\varrho_1^2 &= \frac{1}{r_{12}^2}\Big\{\alpha_{11}(x_1-x_2)^2+\beta_{11}(y_1-y_2)^2+\gamma_{11}(z_1-z_2)^2\Big\},\\ &= \frac{1}{r_{12}^2}\Big\{1(\frac{l}{\sqrt{3}}+\frac{l}{2\sqrt{3}})^2+0(0-\frac{l}{2})^2+0(0-0)^2\Big\},\\ &= \frac{3l^2}{4r_{12}^2},\end{aligned}$$

similarly, we have:

$$\varrho_1^2 = \acute{\varrho_1}^2 = \frac{3l^2}{4r_{12}^2}, \quad \varrho_2^2 = \acute{\varrho_2}^2 = \frac{l^2}{4r_{12}^2}, \quad \varrho_3^2 = \acute{\varrho_3}^2 = 0, \tag{1}$$

where

$$r_{ij} = \sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2}, i \neq j, i = j = 1, 2, 3.$$

The potential V between two triaxial rigid bodies m_1 and m_2 is

$$\begin{aligned}-V &= \frac{Gm_1m_2}{r_{12}}+\frac{Gm_2}{r_{12}^3}\Big[\frac{1}{2}(A_1+B_1+C_1)-\frac{3}{2}(A_1\varrho_1^2+B_1\varrho_1^2\\ &\quad +C_1\varrho_1^2)\Big]+\frac{Gm_1}{r_{12}^3}\Big[\frac{1}{2}(A_2+B_2+C_2)-\frac{3}{2}(A_2\acute{\varrho_1}^2\\ &\quad +B_2\acute{\varrho_1}^2+C_2\acute{\varrho_1}^2)\Big]\text{(Brouwer, Clemence (1961))}\end{aligned} \tag{2}$$

From Eq. (1) and Eq. (2), the potential V becomes

$$-V = \frac{Gm_1m_2}{r_{12}}+\frac{Gm_2}{4r_{12}^3}(\sigma_1+3\sigma_2), \tag{3}$$

where

$$\sigma_1 = \frac{a^2-b^2}{5}, \sigma_2 = \frac{a^2-c^2}{5},$$

are triaxiality parameters.

Since the primaries are moving in circular orbits about their center of mass, the

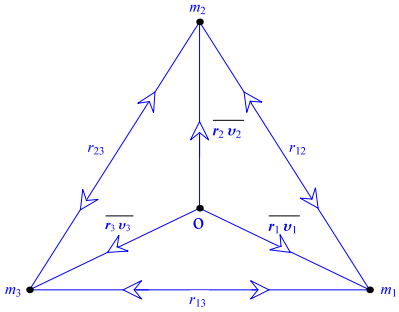


Figure 1 The configuration of three-body problem.

equations of motion of m_1, m_2 and m_3 are

$$\begin{aligned} -n^2 r_1 \hat{v}_1 m_1 &= \left[\frac{Gm_1m_2}{r_{12}^3} + \frac{3Gm_1m_2}{4r_{12}^5}(\sigma_1 + 3\sigma_2)\right](r_2\hat{v}_2 - r_1\hat{v}_1) \\ &+ \left[\frac{Gm_1m_3}{r_{13}^3} + \frac{3Gm_1m_3}{4r_{13}^5}(\sigma_1 + 3\sigma_2)\right](r_3\hat{v}_3 - r_1\hat{v}_1), \end{aligned} \tag{4}$$

$$\begin{aligned} -n^2 r_1 \hat{v}_1 m_2 &= \left[\frac{Gm_2m_3}{r_{23}^3} + \frac{3Gm_2m_3}{4r_{23}^5}(\sigma_1 + 3\sigma_2)\right](r_3\hat{v}_3 - r_2\hat{v}_2) \\ &- \left[\frac{Gm_2m_1}{r_{21}^3} + \frac{3Gm_2m_1}{4r_{21}^5}(\sigma_1 + 3\sigma_2)\right](r_2\hat{v}_2 - r_1\hat{v}_1), \end{aligned} \tag{5}$$

$$\begin{aligned} -n^2 r_3 \hat{v}_3 m_3 &= \left[\frac{Gm_3m_1}{r_{31}^3} + \frac{3Gm_3m_1}{4r_{311}^5}(\sigma_1 + 3\sigma_2)\right](r_3\hat{v}_3 - r_1\hat{v}_1) \\ &- \left[\frac{Gm_2m_1}{r_{21}^3} + \frac{3Gm_3m_2}{4r_{32}^5}(\sigma_1 + 3\sigma_2)\right](r_3\hat{v}_3 - r_2\hat{v}_2), \end{aligned} \tag{6}$$

Since O is the center of mass of m_i $(i = 1, 2, 3)$ therefore we have

$$m_1 r_1 \hat{v}_1 + m_2 r_2 \hat{v}_2 + m_3 r_3 \hat{v}_3 = 0. \tag{7}$$

We further observe that by adding Eq. (4) and Eq. (5), we obtain an equation which, upon substitution from Eq. (7), is Eq. (5). Hence we can use Eq. (7) in place of Eq. (5) as a third equation in the units vectors $\hat{v}_1, \hat{v}_1$, and $\hat{v}_3$. The rectangular coordinates system with origin at O is rotating with angular speed n in the counter clockwise direction. In this system, the unit vectors $\hat{v}_s (s = 1, 2, 3)$ are fixed in position. Let the orientations relative to these cartesian axes be denoted by the angles in equations $\hat{v}_i = \cos\theta_i \mathbf{i} + \sin\theta_i \mathbf{j}$. Thus, Eq. (4), Eq. (5) and Eq. (7) in the cartesian form are

written as:

$$\Big[\quad -n^2 + \frac{Gm_2}{r_{12}^3} + \frac{3Gm_2}{4r_{12}^5}(\sigma_1 + 3\sigma_2) + \frac{Gm_3}{r_{13}^3} + \frac{3Gm_3}{4r_{13}^5}(\sigma_1 + 3\sigma_2)\Big]x_1 - \Big[\frac{Gm_2}{r_{12}^3} + \frac{3Gm_2}{4r_{12}^5}(\sigma_1 + 3\sigma_2)\Big]x_2 - \Big[\frac{Gm_3}{r_{13}^3} + \frac{3Gm_3}{4r_{13}^5}(\sigma_1 + 3\sigma_2)\Big]x_3 = 0, \tag{8}$$

$$-\Big[\quad \frac{Gm_1}{r_{21}^3} + \frac{3Gm_1}{4r_{21}^5}(\sigma_1 + 3\sigma_2)\Big]x_1 + \Big[-n^2 + \frac{Gm_3}{r_{23}^3} + \frac{3Gm_3}{4r_{23}^5} \times (\sigma_1 + 3\sigma_2) + \frac{Gm_1}{r_{21}^3} + \frac{3Gm_1}{4r_{21}^5}(\sigma_1 + 3\sigma_2)\Big]x_2 - \Big[\frac{Gm_3}{r_{31}^3} + \frac{3Gm_3}{4r_{31}^5}(\sigma_1 + 3\sigma_2)\Big]x_3 = 0, \tag{9}$$

$$m_1x_1 + m_2x_2 + m_3x_3 = 0, \tag{10}$$

$$\Big[\quad -n^2 + \frac{Gm_2}{r_{12}^3} + \frac{3Gm_2}{4r_{12}^5}(\sigma_1 + 3\sigma_2) + \frac{Gm_3}{r_{13}^3} + \frac{3Gm_3}{4r_{13}^5}(\sigma_1 + 3\sigma_2)\Big]y_1 - \Big[\frac{Gm_2}{r_{12}^3} + \frac{3Gm_2}{4r_{12}^5}(\sigma_1 + 3\sigma_2)\Big]y_2 - \Big[\frac{Gm_3}{r_{13}^3} + \frac{3Gm_3}{4r_{13}^5}(\sigma_1 + 3\sigma_2)\Big]y_3 = 0, \tag{11}$$

$$-\Big[\quad \frac{Gm_1}{r_{21}^3} + \frac{3Gm_1}{4r_{21}^5}(\sigma_1 + 3\sigma_2)\Big]y_1 + \Big[-n^2 + \frac{Gm_3}{r_{23}^3} + \frac{3Gm_3}{4r_{23}^5} \times (\sigma_1 + 3\sigma_2) + \frac{Gm_1}{r_{21}^3} + \frac{3Gm_1}{4r_{21}^5}(\sigma_1 + 3\sigma_2)\Big]y_2 - \Big[\frac{Gm_3}{r_{31}^3} + \frac{3Gm_3}{4r_{31}^5}(\sigma_1 + 3\sigma_2)\Big]y_3 = 0, \tag{12}$$

$$m_1y_1 + m_2y_2 + m_3y_3 = 0. \tag{13}$$

The equilateral triangle configuration remains invariant with time when the primaries situated at the vertices of an equilateral triangle are spherical in shape. But when the primaries situated at the vertices of an equilateral triangle are triaxial rigid bodies, it is possible that, during their motion, the equilateral triangle configuration may change. It is supposed that during the motion at any time: $r_{12} = r_{21} = l + \vartheta_1, r_{13} = r_{31} = l + \vartheta_2, r_{23} = r_{23} = l + \vartheta_3, (\vartheta_1, \vartheta_2, \vartheta_3 \ll 1)$. Thus, using these relations and neglecting the higher order terms in $\vartheta_i, (i = 1, 2, 3)$, Eqs. 8, 9, 11 and 12 become

$$\begin{aligned}
&\Big[\quad -n^2 + \frac{Gm_2}{l^3}\Big(1 - \frac{3\vartheta_1}{l}\Big) + \frac{3Gm_2}{4l^5}\Big(1 - \frac{5\vartheta_1}{l}\Big)(\sigma_1 + 3\sigma_2) \\
&+ \quad \frac{Gm_3}{l^3}\Big(1 - \frac{3\vartheta_2}{l}\Big) + \frac{3Gm_3}{4l^5}\Big(1 - \frac{5\vartheta_2}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]x_1 \\
&- \quad \Big[\frac{Gm_2}{l^3}\Big(1 - \frac{3\vartheta_1}{l}\Big) + \frac{3Gm_2}{4l^5}\Big(1 - \frac{3\vartheta_1}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]x_2 \\
&- \quad \Big[\frac{Gm_3}{l^3}\Big(1 - \frac{3\vartheta_2}{l}\Big) + \frac{3Gm_3}{4l^5}\Big(1 - \frac{3\vartheta_2}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]x_3 = 0,
\end{aligned} \tag{14}$$

$$\begin{aligned}
-\Big[\quad &\frac{Gm_1}{l^3}\Big(1 - \frac{3\vartheta_1}{l}\Big) + \frac{3Gm_1}{4l^5}\Big(1 - \frac{5\vartheta_1}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]x_1 \\
+ \quad &\Big[- n^2 + \frac{Gm_3}{l^3}\Big(1 - \frac{3\vartheta_3}{l}\Big) + \frac{3Gm_3}{4l^5}\Big(1 - \frac{3\vartheta_3}{l}\Big)(\sigma_1 + 3\sigma_2) \\
+ \quad &\frac{Gm_1}{l^3}\Big(1 - \frac{3\vartheta_1}{l}\Big) + \frac{3Gm_1}{4l^5}\Big(1 - \frac{5\vartheta_1}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]x_2 \\
- \quad &\Big[\frac{Gm_3}{l^3}\Big(1 - \frac{3\vartheta_3}{l}\Big) + \frac{3Gm_3}{4l^5}\Big(1 - \frac{3\vartheta_3}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]x_3 = 0,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\Big[\quad -n^2 + \frac{Gm_2}{l^3}\Big(1 - \frac{3\vartheta_1}{l}\Big) + \frac{3Gm_2}{4l^5}\Big(1 - \frac{5\vartheta_1}{l}\Big)(\sigma_1 + 3\sigma_2) \\
&+ \quad \frac{Gm_3}{l^3}\Big(1 - \frac{3\vartheta_2}{l}\Big) + \frac{3Gm_3}{4l^5}\Big(1 - \frac{5\vartheta_2}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]y_1 \\
&- \quad \Big[\frac{Gm_2}{l^3}\Big(1 - \frac{3\vartheta_1}{l}\Big) + \frac{3Gm_2}{4l^5}\Big(1 - \frac{3\vartheta_1}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]y_2 \\
&- \quad \Big[\frac{Gm_3}{l^3}\Big(1 - \frac{3\vartheta_2}{l}\Big) + \frac{3Gm_3}{4l^5}\Big(1 - \frac{3\vartheta_2}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]y_3 = 0,
\end{aligned} \tag{16}$$

$$\begin{aligned}
-\Big[\quad &\frac{Gm_1}{l^3}\Big(1 - \frac{3\vartheta_1}{l}\Big) + \frac{3Gm_1}{4l^5}\Big(1 - \frac{5\vartheta_1}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]y_1 \\
+ \quad &\Big[- n^2 + \frac{Gm_3}{l^3}\Big(1 - \frac{3\vartheta_3}{l}\Big) + \frac{3Gm_3}{4l^5}\Big(1 - \frac{3\vartheta_3}{l}\Big)(\sigma_1 + 3\sigma_2) \\
+ \quad &\frac{Gm_1}{l^3}\Big(1 - \frac{3\vartheta_1}{l}\Big) + \frac{3Gm_1}{4l^5}\Big(1 - \frac{5\vartheta_1}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]y_2 \\
- \quad &\Big[\frac{Gm_3}{l^3}\Big(1 - \frac{3\vartheta_3}{l}\Big) + \frac{3Gm_3}{4l^5}\Big(1 - \frac{3\vartheta_3}{l}\Big)(\sigma_1 + 3\sigma_2)\Big]y_3 = 0,
\end{aligned} \tag{17}$$

Equations 10, 13, 14, 15, 16 and 17 constitute a system of six simultaneous equations in the six unknowns (x_i, y_i) $(i = 1, 2, 3)$. These equations will have non-trivial solutions if the determinant formed by these equations is zero. By taking $m_i = m$, and $\vartheta_i = \vartheta$, and neglecting the higher order terms of $\sigma_1, \sigma_2, \vartheta_i, \vartheta_i\sigma_1, \vartheta_i\sigma_2$, we get

$$\begin{aligned}
&\frac{m^2(3Gm + l^3n^2)^3\{3Gml(-l + 12\vartheta) + l^5n^2 - 9Gm(\sigma_1 + 3\sigma_2)\}}{l^{14}} \\
&= 0.
\end{aligned} \tag{18}$$

Now, we choose the units of time, mass and distances such that $G = 1$, $m = 1/3$ and $l = 1$, then the first factor of Eq. 19 leads to $n^2 = \frac{3Gm}{l^3}$, i.e. $n^2 = 1$. From the second factor of Eq. 19, we get $\vartheta = \frac{1}{4}(\sigma_1 + 3\sigma_2)$. This shows the required condition under which the equilateral triangle configuration remains stable during the motion if the primaries are triaxial rigid bodies. Thus, the side of the equilateral triangle become $1 + \frac{1}{4}(\sigma_1 + 3\sigma_2)$. We have also observed that our results validate the results of Pandey and Ahmed (2013), when $a_i = b_i$.

However, there exist three limiting cases with respect to the values of the mass parameter μ:

- when $\mu = 0$, it corresponds to the rotating Kepler's central force problem with $m_1 = 1$ situated at the origin of the coordinates.
- when $\mu = \frac{1}{2}$, it corresponds to the circular restricted three body problem, with two equal masses $m_1 = m_2 = \frac{1}{2}$, which describes the Copenhagen problem.
- when $\mu = \frac{1}{3}$, it corresponds to the symmetric case of three bodies with equal masses i.e. $m_1 = m_2 = m_3 = m = \frac{1}{3}$.

In this paper, we have discussed the last case by taking the primaries as triaxial rigid bodies.

3 Properties of the mathematical model and equations of motion

In new equilateral triangle configuration when the primaries $m_i, (i = 1, 2, 3)$ situated at the vertices of an equilateral triangle are triaxial rigid bodies, the new co-ordinates of m_i are $P_1(x_1, y_1, z_1) = (\frac{\alpha'}{\sqrt{3}}, 0, 0)$, $P_2(x_2, y_2, z_2) = (-\frac{\alpha'}{2\sqrt{3}}, \frac{\alpha'}{2}, 0)$ and $P_3(x_3, y_3, z_3) = (-\frac{\alpha'}{2\sqrt{3}}, -\frac{\alpha'}{2}, 0)$, with $\alpha' = 1 + \frac{1}{4}(\sigma_1 + 3\sigma_2)$. Let the co-ordinate of infinitesimal mass m_4 be (x, y, z).

In synodic axis, the equation of motion is

$$m_4\Big\{\frac{\partial^2\vec{r}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial\vec{r}}{\partial t} + \frac{\partial\vec{\omega}}{\partial t} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})\Big\} = \vec{F}, \tag{19}$$

where

$$\text{Relative acceleration} = \frac{\partial^2\vec{r}}{\partial t^2} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k},$$
$$\text{Coriolis acceleration} = \vec{\omega} \times \frac{\partial\vec{r}}{\partial t} = -n\dot{y}\vec{i} + n\dot{x}\vec{j},$$
$$\text{Centrifugal acceleration} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -n^2x\vec{i} - n^2y\vec{j},$$
$$\text{Euler's acceleration} = \frac{\partial\vec{\omega}}{\partial t} \times \overrightarrow{r} = 0.$$

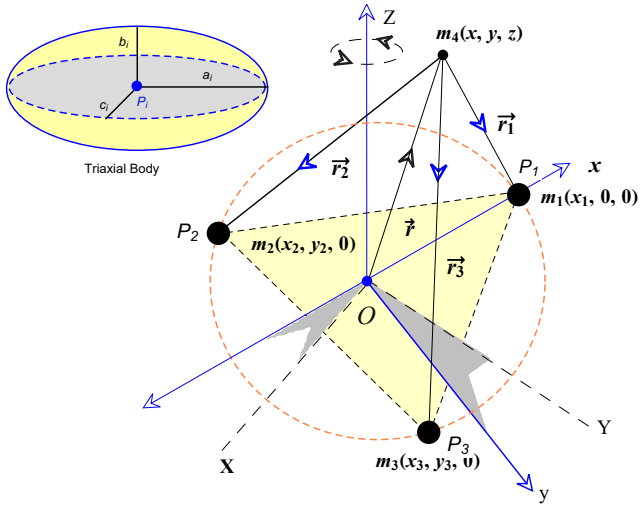


Figure 2 The configuration of planer restricted four-body, where three equally massed primaries are triaxial rigid bodies ($m_1 = m_2 = m_3 = m = \frac{1}{3}$) move on symmetric circular orbits, on the same (x, y) plane. The test particle of negligible mass m_4 is moving under the gravitational influence of the primaries.

We know that the gravitational potential of the irregular bodies P_i, at a point m_4 outside it, is given by

$$V_i = -\frac{Gm_i}{r_i} - \frac{G}{2r_i^3}\Big(I_1 + I_2 + I_3 - 3I\Big) + ..., \tag{20}$$

since a_i, b_i, c_i are the semi-axes of the triaxial rigid bodies of masses m_i. Thus, by Eq. (21) the gravitational potential of third body of mass m_i, at a point m_4 is

$$V = -\frac{Gm_i}{r_i} - \frac{Gm_i}{2r_i^3}\Big\{\sigma_1 + \sigma_2 - \frac{3}{r_i^2}(\sigma_1(y - y_i)^2 + \sigma_2 z^2)\Big\},$$

thus,

$$\begin{aligned}
\vec{F}_i &= -\Big(\frac{\partial V_i}{\partial x}\vec{i} + \frac{\partial V_i}{\partial y}\vec{j} + \frac{\partial V_i}{\partial z}\vec{k}\Big)m_4, \\
&= -Gm_i m_4\Big[(x - x_i)\Big\{\frac{1}{r_i^3} + \frac{3(\sigma_1 + \sigma_2)}{2r_i^5} - \frac{15}{2r_i^7}(\sigma_1(y - y_i)^2 \\
&\quad + \sigma_2 z^2)\Big\}\vec{i} + (y - y_i)\Big\{\frac{1}{r_i^3} + \frac{3(\sigma_1 + \sigma_2)}{2r_i^5} + \frac{3\sigma_1}{r_i^5} - \frac{15}{2r_i^7} \\
&\quad \times(\sigma_1(y - y_i)^2 + \sigma_2 z^2)\Big\}\vec{j} + z\Big\{\frac{1}{r_i^3} + \frac{3(\sigma_1 + \sigma_2)}{2r_i^5} + \frac{3\sigma_2}{r_i^5} \\
&\quad - \frac{15}{2r_i^7}(\sigma_1(y - y_i)^2 + \sigma_2 z^2)\Big\}\vec{k}\Big].
\end{aligned}$$

Thus, the equations of motion of $P_4(m_4$ in non-dimensional co-ordinate system) are

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{21}$$

where

$$\begin{aligned} \Omega &= \frac{1}{2}(x^2+y^2) + \frac{1}{3}\sum_{i=1}^{3}\frac{1}{r_i}\Big\{1 + \frac{(\sigma_1+\sigma_2)}{2r_i^2} - \frac{3}{2r_i^4}\Big(\sigma_1 \\ &\quad \times(y-y_i)^2 + \sigma_2 z^2\Big)\Big\}, \\ r_1 &= \sqrt{\Big(x - \frac{\alpha'}{\sqrt{3}}\Big)^2 + y^2 + z^2}, \\ r_2 &= \sqrt{\Big(x + \frac{\alpha'}{2\sqrt{3}}\Big)^2 + \Big(y - \frac{\alpha'}{2}\Big)^2 + z^2}, \\ r_3 &= \sqrt{\Big(x + \frac{\alpha'}{2\sqrt{3}}\Big)^2 + \Big(y + \frac{\alpha'}{2}\Big)^2 + z^2}, \\ \alpha' &= 1 + \frac{1}{4}(\sigma_1 + 3\sigma_2), \\ \sigma_1 &= \frac{a^2-b^2}{5R^2}, \\ \sigma_2 &= \frac{a^2-c^2}{5R^2}, \quad 0 < \sigma_1, \sigma_2 << 1, \\ n &= 1, \\ R &= \text{the dimensional distance between the primaries.} \end{aligned}$$

Here, Ω_x, Ω_y and Ω_z are the partial derivative of Ω with respect to x, y and z, respectively and "dot" represent differentiation with respect to time. Equations (21) admit the well known Jacobi integral

$$J(x,y,z,\dot{x},\dot{y},\dot{z}) = 2\Omega(x,y,z) - (\dot{x}^2+\dot{y}^2+\dot{z}^2) = C, \tag{22}$$

where $\dot{x},\dot{y},\dot{z}$ are velocities and C is Jacobian constant which is conserved and defines a three-dimensional invariant manifold. The orbit with a given value of it's energy integral is restricted in its motion to regions in which $C \geq 2\Omega(x,y)$, and all other regions are prohibited to the third particle. The value of the energy h is related with the Jacobian constant by $C = -2h$.

4 Discussion and conclusion

In this paper, we have discussed the equilateral restricted four-body problem (ER4BP) formulated on the basis of Lagrange's triangular solutions. First, we have reviewed the

literature regarding four-body problem. In this paper when all the primaries are triaxial rigid bodies, the equilateral triangle configuration no longer exists. So, we have obtained the relation between the side of the equilateral triangle configuration and triaxiality parameters. Thus, the equilateral triangle configuration remains invariant under the obtained relation. Finally we have presented the equations of motion of the infinitesimal mass for the mentioned setup.

Compliance with ethical standards

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