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Formation and stationary state of a spindle-shaped collisionless gravitating system

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Possible scenario of formation and analytic solution of Vlasov equation for the final state of a self-gravitating collisionless system are proposed. The system is assumed to possess axial symmetry and zero total kinetic energy in the initial state. It is shown that under these initial conditions evolution of the velocity distribution goes a special way and after all finalizes with formation of phase density of special form. It turns out that for this form of the phase density the Vlasov equation is solvable. A particular solution is obtained which describes a spindle-shaped cluster of gravitating particles.

Keywords: Axisymmetric collisionless system, analytic solution

1 Introduction

A collisionless system whose particles stay at rest at some initial moment of time, starts to change in shape and density under the action of gravitational forces and after some period of evolution, all changes slow down as the system approaches its equilibrium state. Symmetry of the initial state, particularly, axial symmetry predetermines some properties of the equilibrium state which theoretically is the final of evolution of the system. In this work we discuss the formation of the final state of an initially cold axially-symmetric collisionless self-gravitating system. The goal is to obtain an analytic solution of the Vlasov equation for the state formed under these conditions. In the following considerations we assume that there are no instabilities which break the axial symmetry, hence its gravitational potential remains axially symmetric.

It is natural to consider the system in round cylinder coordinates $\{z, \rho, \varphi\}$. If the motion of each particle of the system starts from the resting state, then accelerates strictly in the half-plane of constant azimuthal angle φ and φ-component of its velocity remains zero forever. Then, during the whole period of evolution and in the final state all particles of the system move only in half-planes $\varphi = \text{const}$ and

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its distribution is extremely flattened in the momentum space. In other words, this distribution is actually two-dimensional due to the factor $\delta(L)$, where L stands for angular momentum of the particle with respect to the axis of symmetry. This form of the distribution function is part of the initial state, which is conserved until the system reaches its final state that simplifies both the analysis of the process of evolution and the construction of the solution of the Vlasov equation.

If the gravitational potential of the system remains axially symmetric, it suffices to consider the evolution of the distribution function only in the planes $L = 0$ in the momentum space, therefore hereafter we term it as "momentum plane". Evolution of the distribution function passes under the influence of two main factors. First, the gravitational potential of the system depends on time, but it does not look possible to solve the Vlasov equation under this condition. Second, the distribution function changes in a special way disregarding of the change of the potential. The point is that each particle obligatorily crosses the axis of symmetry and every time it does, ρ-component of its velocity changes the sign. As a result, the velocity distrubution of an ensemble of particles which cross the axis in a given point at a given moment of time at each point of the axis, but everytime the ensemble consists of particles which before did it in other ensembles and at other points. Hence, the velocity distribution continuously undergoes a mirror reflection on the axis that finally makes it rotationally symmetric. Therefore, disregarding all of the rest, the distribution function takes the form

$$f = \delta(L) f(E), \tag{1}$$

where E stands for energy of particles with zero angular momentum L. So, to describe the final state of the system it suffices to solve the Vlasov equation for the distribution of this form.

2 Integration over velocities

Integration of the distribution function over velocities is a necessary part of solving the Vlasov equation. Usually the velocity space is considered as uniquely given one for each point of the space. In fact, velocities in a given point P belong to space T_P tangent to the physical space in this point, hence, we have one velocity space for each point. Since the physical space is taken as Euclidean one, there is no reason to introduce so-called tangent bundle T_M over the space (here M stands for the space). On the other hand, one of integrals of motion namely, L contains only one component of velocity $\dot{\varphi}$ which has a different direction at different points, therefore it is much more convenient to integrate over it using different local frames everywhere. That is why it is convenient to follow N. Chernikov (Chernikov, 1963) and use the idea of tangent bundle when integrating over velocities. According to this, the Vlasov equation takes the form

$$\mu(z, \rho) \equiv \int\limits_{T_p} \mathrm{d}\vec{v} \delta(L) f(E), \tag{2}$$

where μ and $\mathrm{d}\vec{v}$ stand for mass density and the measure of integration respectively.

Each particle has velocity with components $\dot{z}$ and $\dot{\rho}$ while its $\dot{\varphi}$ component is identically zero. The total energy of a particle is

$$E = \frac{1}{2}(\dot{z}^2 + \dot{\rho}^2) + \Phi, \quad \Phi < 0,$$

this inequality specifies the domain of integration of the distribution. The measure of integration is

$$\mathrm{d}\vec{v} = \mathrm{d}\dot{z}\,\mathrm{d}\dot{\rho}\,\rho\,\mathrm{d}\dot{\varphi} = \mathrm{d}\dot{z}\,\mathrm{d}\dot{\rho}\,\rho^{-1}\,\mathrm{d}L_z.$$

Substituting this into the integral in the right-hand side of the equation (2) yields

$$\mu = \int\limits_{-\infty}^{\infty} \mathrm{d}\dot{z} \int\limits_{-\infty}^{\infty} \mathrm{d}\dot{\rho} \int\limits_{-\infty}^{\infty} \rho^{-1}\,\mathrm{d}L_z\, f(-2E)\delta(L) =$$

$$= \frac{1}{\rho} \int\limits_{-\infty}^{\infty} \mathrm{d}\dot{z} \int\limits_{-\infty}^{\infty} \mathrm{d}\dot{\rho}\, f(-2\Phi - \dot{z}^2 - \dot{\rho}^2),$$

where the domain of integration is restricted with the condition $E \leq 0$. Since the integrand has rotational symmetry in the velocity space it is convenient to consider it in polar coordinates $\{v, \alpha\}$ where v stands for the absolute value of the particle velocity and α for the angle between the vector of velocity and the z-axis. Therefore, the measure of integration becomes $\mathrm{d}\vec{v} = v\,\mathrm{d}v\,\mathrm{d}\alpha$ and mass density takes the form

$$\mu = \frac{1}{\rho} \int\limits_{0}^{\sqrt{-2\Phi - v^2}} v\,\mathrm{d}v \int\limits_{0}^{2\pi} \mathrm{d}\alpha\, f(-2\Phi - \dot{z}^2 - \dot{\rho}^2).$$

Since the integrand does not depend on the angle, integration over the variable α is trivial, so, finally we have

$$\mu = \frac{2\pi}{\rho} \int\limits_{0}^{\sqrt{-2\Phi - v^2}} v\,\mathrm{d}v f(-2\Phi - v^2).$$

Evidently, whatever form the function to be found will have, the integral depends only on the value of the gravitational potential in the point. Consequently, the mass density has the form

$$\mu = \frac{F(\Phi)}{\rho} \tag{3}$$

where $F(\Phi)$ is an unknown function of the gravitational potential Φ, which depends on the explicit form of the function $F(-2E)$.

3 The gravitational potential

Correspondingly, the Poisson equation turns into a non-linear equation for the gravitational potential with the function (3) as the right-hand side:

$$\triangle\Phi = -\frac{4\pi F(\Phi)}{\rho}. \tag{4}$$

Therefore, the next task is to find out a relevant solution of this equation.

Though, in general, it is hard to solve non-linear equations like the equation (4), one simple solution can be found easily. One simple solution of this equation has the form

$$\Phi(u,v) = -\frac{m}{\sqrt{z^2+(\rho+a)^2}},$$

where a is an arbitrary constant of dimension of length. Indeed, differentiating the potential gives:

$$\triangle\Phi = \frac{am}{\rho[z^2+(\rho+a)^2]^{3/2}} = -\frac{a\Phi^3}{m^2\rho} \tag{5}$$

that corresponds to the equation (3) with function $F(\Phi) = a\Phi^3/m^2$. Let us now consider the physical properties of this potential.

Conclusion

The model constructed above is based on two main assumptions. The first of them reads that initially all particles of a collisionless system are at rest. If the future object is formed by gravitational contraction of highly rarified matter and its particles pass huge distances of acceleration, their initial velocities are negligibly small compared with velocities they have after formation of the object. Therefore, it is safe to accept this assumption. Another assumption reads that initially the cloud possesses axial symmetry. So, the key points of our construction are:

- if the system is initially cold and possesses axial symmetry, velocity distribution of its particles is highly flattened onto the planes $\dot{\varphi} = 0$ in the velocity space;
- disregarding all of the rest, this distribution rounds at every point due to continuous turning of the distribution into its mirror reflection so that finally it takes the form (1).

All the rest is made by means of standard mathematical methods. The idea of constructing the form of gravitational potential is similar to that used by G.G. Kuzmin in his work (Kuzmin, 1956).

References

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